

**ELEMENTARY PARTICLE
INTERACTIONS
IN A BACKGROUND
MAGNETIC FIELD**

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KAUSHIK BHATTACHARYA
THEORY GROUP
SAHA INSTITUTE OF NUCLEAR PHYSICS
KOLKATA

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CHAPTER 1

Introduction and overview

1.1 A brief overview on magnetic fields in the cosmos, their observation, their origin and particle physics

Magnetic fields are most frequently found in various length scales and various positions of our universe. The average magnitude of the Earth's magnetic field is about 10^{-1}G . This magnetic field encircles the Earth and produces a magnetosphere surrounding it. The field is not uniform. Charged elementary particles from the cosmos sometimes gets trapped in this magnetic field and hover over our Earth in a belt shaped region which is popularly called the Van Allen belt. Also the solar storms produce electromagnetic disturbances on the magnetic field of the Earth and the particles trapped in the Van Allen belt, resulting in ‘aurora borealis’, one of the strangest things seen by human beings. In the solar black spots magnetic fields can go up to 10^3G . Very high magnetic fields are found in the cores of supernovas, and this magnetic field remains frozen on the proto-neutron star remnant whose surface magnetic field has been measured to be 10^{13}G . Not only the planets and the stars have magnetic fields associated with them there are magnetic fields in the galactic and intergalactic mediums. In Milky Way the average galactic magnetic field is of the order of $3 - 4\text{G}$ with a length scale of a few Kilopersecs. Average intergalactic magnetic fields can be of order 10^{-11}G [1] with a length scale of about 1 Megaparsec. Except gravitational interactions which becomes important at astronomical distances (some Megaparsecs) at average energies (much lesser than the Planck scale 10^{19}Gev) the only other field which is omnipresent in the universe in various shapes, sizes and magnitudes is the magnetic field.

The main observational tracers of galactic and extra-galactic magnetic fields are [1, 2] the Zeeman splitting of spectral lines, the intensity and polarization of synchrotron emission from free relativistic electrons and Faraday rotations of polarized electromagnetic radiation passing through an ionized medium. The Zeeman splitting is too small to be

useful outside our own galaxy. The synchrotron emission method and Faraday rotation measurements require an independent estimate of the local electron density. If the magnetic field to be measured is far away one relies on Faraday rotation. The agreement generally found between the strength of the field determined by Faraday rotation technique and that inferred from the analysis of the synchrotron emission in relatively close objects gives reasonable confidence on the reliability of the first method also for far away systems.

The origin of the magnetic fields observed in the galaxies and in the cluster of galaxies is unknown. This is an outstanding problem in modern astrophysics and historically it was the first motivation to look for a primordial origin of the magnetic fields [3]. The general trend is to use magnetohydrodynamic methods to amplify very weak magnetic fields into the μG fields generally observed in galaxies. Today the efficiency of such a mechanism for production of magnetic fields is in question from new observations of magnetic fields in high redshifted galaxies. Furthermore, even if the magnetohydrodynamical calculations are taken into account, the origin of the seed fields which initiated the process has still to be identified. It is understood that somewhere elementary particle physics will enter the scene to explain the formation of the elusive seed fields. With the cosmological observational evidences, as cosmic micro wave background radiation (CMBR) and nucleosynthesis data, big bang theory is a cosmological reality now and so is standard big bang cosmology. In near future perhaps we will have a standard mechanism for production of the seed fields coming out from big bang cosmology.

In the previous paragraph we discussed how models of particle physics can be used to produce the seed fields. But the magnetic fields thus produced can themselves affect various elementary particle interactions. Magnetic fields can enhance the scattering cross-sections or decay rates from their vacuum values. The action of magnetic fields on elementary particle interactions can produce non trivial results and can explain some of the interesting observations in astrophysics. The main portion of this thesis deals with elementary particle interactions, of which the neutrino is of prime concern, in presence of a uniform background magnetic field. To put the theme in perspective the following section gives an overview of the nature of work done in this field by previous workers and the main points around which this thesis evolves.

1.2 Some well known observations, results and motivation for this thesis

Neutrinos have no electric charge. So they do not have any direct coupling to photons in any renormalizable quantum field theory. The standard Dirac contribution to the magnetic moment, which comes from the vector coupling of a fermion to the photon, is therefore absent for the neutrino. In the standard model of electroweak interactions, the neutrinos cannot have any anomalous magnetic moment either. The reason is: anomalous magnetic moment comes from chirality-flipping interactions $\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$, and neutrinos cannot have such interactions because there are no right-chiral neutrinos in the standard model. Other particles as the electron and muon have normal and anomalous magnetic moments and so can interact with external magnetic fields.

Inclusion of neutrino mass naturally takes us beyond the standard model, where the issue of neutrino interactions with a magnetic field must be reassessed. If the massive neutrino turns out to be a Dirac fermion, its right-chiral projection must be included in the fermion content of the theory, and in that case an anomalous magnetic moment of a neutrino automatically emerges when quantum corrections are taken into account. In the simplest extension of the standard model including right-chiral neutrinos, the magnetic moment arises from the diagrams in Fig. 1.1 and is given by [4]

$$\mu_\nu = \frac{3eG_F m_\nu}{8\sqrt{2}\pi^2} = 3 \times 10^{-19} \mu_B \times \left(\frac{m_\nu}{1 \text{ eV}} \right), \quad (1.1)$$

where m_ν is the mass of the neutrino and μ_B is the Bohr magneton. If, on the other hand, neutrinos have Majorana masses,¹ i.e., they are their own antiparticles, they cannot have any magnetic moment at all, because **CPT** symmetry implies that the magnetic moments of a particle and its antiparticle should be equal and opposite. However, even in this case there can be transition magnetic moments, which are co-efficients of effective operators of the form $\bar{\psi}_1\sigma_{\mu\nu}\psi_2 F^{\mu\nu}$, where ψ_1 and ψ_2 denote two different fermion fields. These will also indicate some sort of interaction with the magnetic field, associated with a change of the fermion flavour. The question of the neutrino magnetic moment assumed importance when it was suggested that it can be a potential solution for the solar neutrino puzzle [6, 7, 8]. A viable solution required a neutrino magnetic moment around $10^{-10} \mu_B$, orders of magnitude larger than that given by Eq. (1.1), knowing that the neutrino masses cannot

¹A detailed discussion on Dirac and Majorana masses of neutrinos is present in [5].

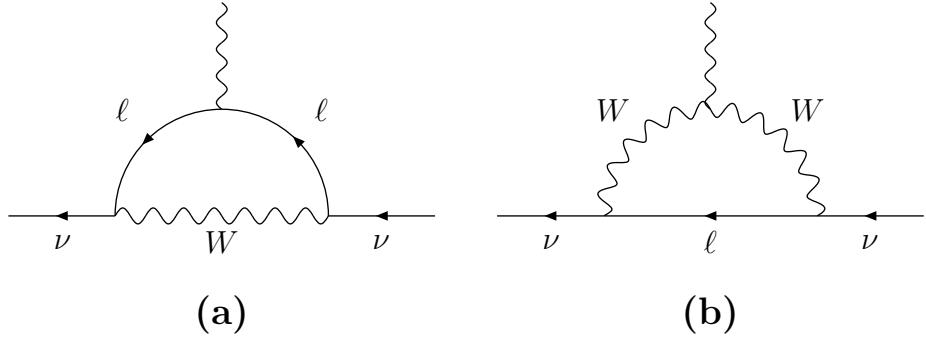


Figure 1.1: One-loop diagrams that give rise to neutrino magnetic moment in standard model aided with right-handed neutrinos. The lines marked ν are generic neutrino lines, whereas those marked ℓ are generic charged leptons. The external vector boson line is the photon. In renormalizable gauges, there are extra diagrams where any of the W lines can be replaced by the corresponding unphysical Higgs scalar.

be very large. At present the situation has changed and neutrino magnetic moment is not a lucrative object to look for to explain the solar neutrino data. The Sudbury neutrino observatory results [9] indicate no deficiency of left handed neutrinos in neutral-current interactions.

If we leave out non-standard neutrinos and take its magnetic moment to be zero, then also magnetic fields can alter their properties. This is the prime concern of the present thesis. One of the simplest things that gets affected is the neutrino self-energy in a magnetic field. The magnetic field enters the scene through interaction with the charged leptons in the loops of the self-energy diagram. Erdas and Feldman calculated the dispersion relation of neutrinos in a uniform background magnetic field. To lowest order in the magnetic field \mathcal{B} they obtained the dispersion relation [10]

$$\Omega^2 = \mathbf{q}^2 + \left(\frac{eg}{2\pi M_W^2} \right)^2 \left(\frac{1}{3} \ln \frac{M_W}{m} + \frac{1}{8} \right) \mathcal{B}^2 q_{\perp}^2. \quad (1.2)$$

Here Ω and \mathbf{q} are the neutrino energy and 3-momentum. q_{\perp} is the magnitude of that component of the neutrino 3-momentum which is perpendicular to the magnetic field 3-vector. m is the mass of the charged fermion which has the same flavour as that of the neutrino, M_W the W boson mass and g the weak coupling constant. The origin of such a term like q_{\perp}^2 will be discussed in chapter 3. The dispersion relation reduces to the usual dispersion relation of a zero mass particle when we put $\mathcal{B} = 0$. For strong magnetic fields,

the dispersion relation has been calculated more recently by Elizalde, Ferrer and de la Incera [11].

There are other processes as neutrino nucleon scattering which are also affected by magnetic fields. The charged current interaction Lagrangian involving neutrinos and nucleons is given by

$$\mathcal{L}_{\text{int}} = \sqrt{2}G_\beta \left[\bar{\psi}_{(e)} \gamma^\mu L \psi_{(\nu_e)} \right] \left[\bar{\psi}_{(p)} \gamma_\mu (G_V + G_A \gamma_5) \psi_{(n)} \right], \quad (1.3)$$

where $L = \frac{1}{2}(1 - \gamma_5)$ and $G_\beta = G_F \cos \theta_C$, θ_C being the Cabibbo angle, and $G_A/G_V = -1.26$. This can be used to find the cross-section for various neutrino-nucleon scattering processes.

The above interaction Lagrangian is most vital for processes involving neutrinos or antineutrinos. In a star, when such reactions occur, the final neutrino or the antineutrino escapes and the star loses energy. Such processes are collectively known as URCA processes, named after a casino in Rio de Janeiro where customers lose money little by little [12]. These processes are,

$$n \rightarrow p + e^- + \bar{\nu}, \quad (1.4)$$

$$p + e^- \rightarrow n + \nu, \quad (1.5)$$

$$n + e^+ \rightarrow p + \bar{\nu}. \quad (1.6)$$

The calculations of the cross-sections and decay rates in presence of a magnetic field uses a benchmark value of the background field:

$$\begin{aligned} \mathcal{B}_e &= \frac{m^2}{e}, \\ &= 4.41 \times 10^{13} \text{G}, \end{aligned} \quad (1.7)$$

where m is the electron mass, and e is the charge of the proton. \mathcal{B}_e is sometimes called the ‘critical field’ although nothing that critical happens at this field strength. The only condition which defines the critical field is that when the magnitude of the magnetic field reaches the value \mathcal{B}_e the electron cyclotron frequency equals its rest mass in the natural units. If the magnitude of the magnetic field is above the critical field the electronic wave functions get considerably modified. It is assumed henceforth that while discussing scattering or decay processes the magnitude of the external fields will be around \mathcal{B}_e . If the magnitude of the field is comparable to m_p^2/e then the proton wave functions can also get affected. The calculations in this thesis will not involve such high fields.

The rate of the neutron beta decay process in a magnetic field was calculated by various authors. An early paper by Fassio-Canuto [13] derived the rate in a background of degenerate electrons. The work considers exact wave functions of the electrons in a background of a uniform magnetic field. Contemporary papers by Matese and O'Connell [14, 15] derived the rate where the background did not contain any matter, but included the effects of the polarization of neutrons due to the magnetic field. Protons and neutrons were assumed to be non-relativistic in the calculations. Further, the magnetic field was assumed to be much smaller than m_p^2/e so that its effect on the proton wave function could be neglected. Various calculations of the other processes exist in the literature. Some calculations take the background matter density into account [16, 17, 18, 19, 20], some include magnetic effects on the proton wavefunction as well [20]. In all of these calculations the cross-section or the decay rate is sensitive to the direction between the neutrino 3-momentum and the external uniform magnetic field.

Neutrinos can also interact with photons, but this interaction is an effective one mediated by virtual charged particles. In a magnetic field the propagators of these charged particles are affected by the external field. Previously it has been shown how the neutrino self-energy is modified in an external magnetic field. The properties of photons also get modified in an external uniform magnetic field due to their interaction with virtual charged particles. It was shown by Adler [21] that when the background magnetic field magnitude is around \mathcal{B}_e many interesting things can happen. The photon dispersion relation in presence of a magnetic field changes from that in vacuum $|\mathbf{k}| = \omega$. There are actually two different indices of refraction corresponding to the two photon propagation modes. The modes are linearly polarized, with the magnetic field of the mode either parallel or perpendicular to the plane containing the external magnetic field and the direction of propagation. Most of the calculations done by Adler uses the effective Euler-Heisenberg Lagrangian [22]. Later Schwinger, while trying to derive the Euler-Heisenberg Lagrangian, derived the form of the propagator of the charged particles in an uniform background magnetic field [23]. After Schwinger's work most of the calculations, as the self-energy of the photon in a magnetic field, are done using the Schwinger propagator.

Two second rank tensors play an important part in building up the effective neutrino photon vertex function. They are the photon self-energy $\Pi_{\mu\nu}$ and another tensor $\Pi_{\mu\nu}^5$. To one loop the tensors $\Pi_{\mu\nu}^5$ and $\Pi_{\mu\nu}$ are similar except one point. In $\Pi_{\mu\nu}$ both the vertices, corresponding to the two tensor indices, are of vector type whereas in $\Pi_{\mu\nu}^5$ one of the

vertices is of the axial-vector type [24]. $\Pi_{\mu\nu}^5$ is called the axialvector-vector amplitude in this thesis. In a magnetic field background $\Pi_{\mu\nu}$ was first calculated by Tsai [25]. In a different work DeRaad, Milton and Hari Dass [26] calculated $\Pi_{\mu\nu}^5$. Both these calculations were to one loop but to all orders in the external magnetic field. Using the expression of the above two tensors in a magnetic field the rate of the neutrino Cherenkov process,

$$\nu(q) \rightarrow \nu(q') + \gamma(k), \quad (1.8)$$

where q , q' and k are the four momenta of the particles as shown, was calculated by Raffelt and Ioannisian [27, 28]. The neutrino Cherenkov process is kinematically forbidden in vacuum. But in a magnetic field as the photon dispersion relation changes so this process becomes viable. The photon self-energy $\Pi_{\mu\nu}$ [29, 30] and $\Pi_{\mu\nu}^5$ [31, 32, 33] both have been calculated in a magnetized medium to one loop.

In this thesis the main emphasis has been given on neutrino scattering processes in presence of a magnetic field and the electromagnetic vertex of neutrinos in a magnetized medium. $\Pi_{\mu\nu}^5$, which is related to the electromagnetic vertex of neutrinos, has been calculated in the background of a magnetized medium. Using the expression of $\Pi_{\mu\nu}^5$ attempt has been made to find out a specific limit of this quantity, which gives the ‘effective electric charge’ of the neutrino. A general form factor analysis of $\Pi_{\mu\nu}^5$ for various backgrounds, as in vacuum, thermal medium, magnetic field and a magnetized medium is also given in chapter 4. Neutrino interactions in presence of a classical background magnetic field is a subject which has been studied for a long time and most of the topics mentioned above are briefly discussed in the review by Bhattacharya and Pal [34].

1.3 Notations and conventions

Before going into the second chapter, this section summarizes the mathematical notations and conventions we use throughout this thesis. In future the magnitude of the uniform background magnetic field will always be denoted by the symbol \mathcal{B} and the magnetic field vector pointing in the z direction will be given by

$$\mathbf{B} = \hat{z}\mathcal{B}, \quad (1.9)$$

where \hat{z} is the unit vector along the z -axis.

The first thing to note is that a uniform classical magnetic field chooses a certain preferred direction in space. Consequently the Lorentz invariance of the system is restricted,

arbitrary boosts will not preserve a pure magnetic field. A boost along the magnetic field direction or a rotation about the external field direction, are the only Lorentz transformations allowed now. As a result the 4-vector structure breaks down into a perpendicular part and a parallel part. If a^μ is a 4-vector, then

$$a_\parallel^\mu = (a^0, 0, 0, a^3), \quad (1.10)$$

$$a_\perp^\mu = (0, a^1, a^2, 0), \quad (1.11)$$

such that

$$a^\mu = a_\parallel^\mu + a_\perp^\mu. \quad (1.12)$$

Also in our convention

$$g_{\mu\nu} = g_{\mu\nu}^{\parallel} + g_{\mu\nu}^{\perp}, \quad (1.13)$$

where

$$g_{\mu\nu}^{\parallel} = (1, 0, 0, -1), \quad (1.14)$$

$$g_{\mu\nu}^{\perp} = (0, -1, -1, 0). \quad (1.15)$$

Some times instead of a_\parallel^μ and a_\perp^μ use has been made of $a^{\mu\parallel}$ and $a^{\mu\perp}$, they are the same things written in two different ways.

Frequently we will come across terms as a_\parallel^2 and a_\perp^2 . These terms stands for

$$a_\parallel^2 = g_{\mu\nu}^{\parallel} a^\mu a^\nu, \quad (1.16)$$

$$= (a^1)^2 - (a^3)^2, \quad (1.17)$$

$$a_\perp^2 = -g_{\mu\nu}^{\perp} a^\mu a^\nu, \quad (1.18)$$

$$= (a^1)^2 + (a^2)^2, \quad (1.19)$$

such that

$$a^2 = a_\parallel^2 - a_\perp^2. \quad (1.20)$$

Sometimes the equations will involve terms as $(a \cdot \tilde{b})_{\parallel}$, where a and b are two 4-vectors. This dot product stands for

$$(a \cdot \tilde{b})_{\parallel} = (a^0 b^3 - a^3 b^0) = -(a_0 b_3 - a_3 b_0), \quad (1.21)$$

which implies

$$(a \cdot \tilde{b})_{\parallel} = -(b \cdot \tilde{a})_{\parallel},$$

and as a result

$$(a \cdot \tilde{a})_{\parallel} = 0.$$

1.4 Scheme of things to come

This last section is about the scheme of the things to come. Chapter 2 focuses on the quantum field theoretical description of scattering processes which involves charged particles, in presence of a background uniform magnetic field. The field theoretical techniques are then applied to find out the cross-section of the inverse beta decay process. Using the expression of the inverse beta decay cross-section an attempt has been made to explain the puzzle regarding the high velocities of pulsars.

Chapter 3 discusses about the charged particle propagators in an uniform background magnetic field using Schwinger's method [23]. The Schwinger propagator is introduced and then by using the methods of quantum statistical field theory it is expressed in a thermal background. A general analysis on neutrino self-energy in a magnetic field and in a magnetized medium follows. The chapter ends with a discussion on the effects of a background magnetic field on neutrino oscillation phenomenology.

Chapter 4 deals with the topic of neutrino photon interactions. It is shown that the photon vacuum polarization tensor $\Pi_{\mu\nu}$ and the axialvector-vector amplitude $\Pi_{\mu\nu}^5$ are essential ingredients of the neutrino electromagnetic vertex. Some comments on $\Pi_{\mu\nu}$ and a discussion on $\Pi_{\mu\nu}^5$ is presented in this chapter. The chapter ends with an expression of the effective electric charge of neutrinos in a magnetized medium.

The last chapter concludes by summarizing the points which are discussed in the various chapters of this thesis. The thesis ends with a set of appendices containing some detailed calculations which were put at the end so as not to block the readers mind with lengthy calculations.

CHAPTER 2

Inverse beta decay in a constant background magnetic field

2.1 Introduction

The interactions of elementary particles show novel features when they occur in non-trivial backgrounds. Study of particle propagation in matter has proved pivotal in the understanding of the solar neutrino problem. A typical example of this is the neutrino-electron scattering in presence of a medium which culminated in the now famous MSW [35, 36] effect. The cross-section for the elastic neutrino-electron scattering in presence of a magnetic field has been studied [37]. Bezchastnov and Haensel [38] has also calculated the neutrino-electron scattering cross-section in presence of a magnetized medium.

There are processes related to the neutrino-electron scattering, obtained by crossing, which contain a neutrino-antineutrino pair in the final state. For example, one can have the pair annihilation of electron and positron into neutrino-antineutrino pair, or the neutrino synchrotron radiation:

$$e^- \rightarrow e^- + \nu + \bar{\nu}. \quad (2.1)$$

This is similar to a the normal synchrotron radiation reaction, the difference being that a neutrino-antineutrino pair is produced instead of a photon. It should be noted that this process cannot occur in the vacuum. However, in the presence of a background magnetic field and background matter, the dispersion relation of the electron changes so that it becomes kinematically feasible. These processes provide important mechanism for stellar energy loss, and the rates of these processes have been calculated [39, 40, 41]. The reverse of these processes are important for neutrino absorption, and have also been studied [42]. Moreover studies of scattering cross-sections and decay rates in background magnetic

fields are important since stellar objects like neutron stars are expected to possess very high magnetic fields, of the order of 10^{12} G or higher. Analysis of these processes might be crucial for obtaining a proper understanding of the properties of these stars. As the preferred direction of the background magnetic fields break the isotropy of space the scattering cross-sections of various processes calculated in such a background also becomes anisotropic. This anisotropy of the cross-sections of processes involving neutrinos as initial particles or final particles can give rise to an overall asymmetric emission of neutrinos from a newly born neutron star and as a result the star may get an overall momentum thrust.

As discussed in chapter 1, there are various possible ways to introduce the external uniform magnetic field in the calculation of processes including elementary particles. In this chapter we explore the exact solutions of the Dirac equation in the presence of such an external field, and then apply the results to obtain the physically interesting numbers as scattering cross-sections [43, 37]. The initial sections of this chapter develops a consistent quantum field theoretical method for calculations of cross-sections in a background magnetic field. The discussions up to section 2.3 is general, the results discussed can be appropriately used in any scattering cross-section calculation in a background magnetic field. The particularities of the inverse beta decay process cross-section calculation enters from section 2.4. Based upon the form of the scattering cross-section, section 2.5 elucidates the point about asymmetric emission of neutrinos from a proto-neutron star. The chapter concludes by summarizing the ideas exposed.

2.2 Solutions of the Dirac equation in a uniform magnetic field

For a particle of mass m and charge eQ , the Dirac equation in presence of a magnetic field is given by

$$i\frac{\partial\psi}{\partial t} = [\boldsymbol{\alpha} \cdot (-i\nabla - eQ\mathbf{A}) + \beta m]\psi, \quad (2.2)$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices, and \mathbf{A} is the vector potential. In our convention, e is the positive unit of charge, taken as usual to be equal to the proton charge.

For stationary states, we can write

$$\psi = e^{-iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad (2.3)$$

where ϕ and χ are 2-component objects. We use the Pauli-Dirac representation of the Dirac matrices, in which

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.4)$$

where each block represents a 2×2 matrix, and $\boldsymbol{\sigma}$ are the Pauli matrices. With this notation, we can write Eq. (2.2) as

$$(E - m)\phi = \boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} - eQ\mathbf{A})\chi, \quad (2.5)$$

$$(E + m)\chi = \boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} - eQ\mathbf{A})\phi. \quad (2.6)$$

Eliminating χ , we obtain

$$(E^2 - m^2)\phi = [\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} - eQ\mathbf{A})]^2 \phi. \quad (2.7)$$

We will work with a constant magnetic field \mathcal{B} . Without loss of generality, it can be taken along the z -direction. The vector potential can be chosen in many equivalent ways. We take

$$A_0 = A_y = A_z = 0, \quad A_x = -y\mathcal{B}. \quad (2.8)$$

With this choice, Eq. (2.7) reduces to the form

$$(E^2 - m^2)\phi = \left[-\boldsymbol{\nabla}^2 + (eQ\mathcal{B})^2 y^2 - eQ\mathcal{B}(2iy\frac{\partial}{\partial x} + \sigma_3) \right] \phi. \quad (2.9)$$

Here σ_3 is the diagonal Pauli matrix. Noticing that the co-ordinates x and z do not appear in the equation except through the derivatives, we can write the solutions as

$$\phi = e^{i\mathbf{p} \cdot \mathbf{X}_y} f(y), \quad (2.10)$$

where $f(y)$ is a 2-component matrix which depends only on the y -coordinate, and possibly some momentum components, as we will see shortly. We have also introduced the notation \mathbf{X} for the spatial co-ordinates (in order to distinguish it from x , which is one of the components of \mathbf{X}), and \mathbf{X}_y for the vector \mathbf{X} with its y -component set equal to zero. In other words, $\mathbf{p} \cdot \mathbf{X}_y \equiv p_x x + p_z z$, where p_x and p_z denote the eigenvalues of momentum in the x and z directions.²

²It is to be understood that whenever we write the spatial component of any vector with a lettered subscript, it would imply the corresponding contravariant component of the relevant 4-vector.

There will be two independent solutions for $f(y)$, which can be taken, without any loss of generality, to be the eigenstates of σ_3 with eigenvalues $s = \pm 1$. This means that we choose the two independent solutions in the form

$$f_+(y) = \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix}, \quad f_-(y) = \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix}. \quad (2.11)$$

Since $\sigma_3 f_s = s f_s$, the differential equations satisfied by F_s is

$$\frac{d^2 F_s}{dy^2} - (eQ\mathcal{B}y + p_x)^2 F_s + (E^2 - m^2 - p_z^2 + eQ\mathcal{B}s)F_s = 0, \quad (2.12)$$

which is obtained from Eq. (2.9). The solution is obtained by using the dimensionless variable

$$\xi = \sqrt{e|Q|\mathcal{B}} \left(y + \frac{p_x}{eQ\mathcal{B}} \right), \quad (2.13)$$

which transforms Eq. (2.12) to the form

$$\left[\frac{d^2}{d\xi^2} - \xi^2 + a_s \right] F_s = 0, \quad (2.14)$$

where

$$a_s = \frac{E^2 - m^2 - p_z^2 + eQ\mathcal{B}s}{e|Q|\mathcal{B}}. \quad (2.15)$$

This is a special form of Hermite's equation, and the solutions exist provided $a_s = 2\nu + 1$ for $\nu = 0, 1, 2, \dots$. This provides the energy eigenvalues

$$E^2 = m^2 + p_z^2 + (2\nu + 1)e|Q|\mathcal{B} - eQ\mathcal{B}s, \quad (2.16)$$

and the solutions for F_s are

$$N_\nu e^{-\xi^2/2} H_\nu(\xi) \equiv I_\nu(\xi), \quad (2.17)$$

where H_ν are Hermite polynomials of order ν , and N_ν are normalizations which we take to be

$$N_\nu = \left(\frac{\sqrt{e|Q|\mathcal{B}}}{\nu! 2^\nu \sqrt{\pi}} \right)^{1/2}. \quad (2.18)$$

We stress that the choice of normalization can be arbitrarily made, as will be clarified later. With our choice, the functions I_ν satisfy the completeness relation

$$\sum_\nu I_\nu(\xi) I_\nu(\xi_*) = \sqrt{e|Q|\mathcal{B}} \delta(\xi - \xi_*) = \delta(y - y_*), \quad (2.19)$$

where ξ_\star is obtained by replacing y by y_\star in Eq. (2.13).

So far, Q was arbitrary. We now specialize to the case of electrons, for which $Q = -1$. The solutions are then conveniently classified by the energy eigenvalues

$$E_n^2 = m^2 + p_z^2 + 2ne\mathcal{B}, \quad (2.20)$$

which is the relativistic form of Landau energy levels. The solutions are two fold degenerate in general: for $s = 1$, $\nu = n - 1$ and for $s = -1$, $\nu = n$. In the case of $n = 0$, only the second solution is available since ν cannot be negative. The solutions can have positive or negative energies. We will denote the positive square root of the right side by E_n . Representing the solution corresponding to this n -th Landau level by a superscript n , we can then write for the positive energy solutions,

$$f_+^{(n)}(y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix}, \quad f_-^{(n)}(y) = \begin{pmatrix} 0 \\ I_n(\xi) \end{pmatrix}. \quad (2.21)$$

For $n = 0$, the solution f_+ does not exist. We will consistently incorporate this fact by defining

$$I_{-1}(y) = 0, \quad (2.22)$$

in addition to the definition of I_n in Eq. (2.17) for non-negative integers n .

The solutions in Eq. (2.21) determine the upper components of the spinors through Eq. (2.10). The lower components, denoted by χ earlier, can be solved using Eq. (2.6), and finally the positive energy solutions of the Dirac equation can be written as

$$e^{-ip \cdot X_y} U_s(y, n, \mathbf{p}_y), \quad (2.23)$$

where X^μ denotes the space-time coordinate. And U_s are given by

$$U_+(y, n, \mathbf{p}_y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z}{E_n + m} I_{n-1}(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{E_n + m} I_n(\xi) \end{pmatrix}, \quad U_-(y, n, \mathbf{p}_y) = \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{E_n + m} I_{n-1}(\xi) \\ -\frac{p_z}{E_n + m} I_n(\xi) \end{pmatrix}. \quad (2.24)$$

A similar procedure can be adopted for negative energy spinors which have energy eigenvalues $E = -E_n$. In this case, it is easier to start with the two lower components first and then find the upper components from Eq. (2.5). The solutions are

$$e^{ip \cdot X_y} V_s(y, n, \mathbf{p}_y), \quad (2.25)$$

where

$$V_+(y, n, \mathbf{p}_y) = \begin{pmatrix} \frac{p_z}{E_n + m} I_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2ne\mathcal{B}}}{E_n + m} I_n(\tilde{\xi}) \\ I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix}, \quad V_-(y, n, \mathbf{p}_y) = \begin{pmatrix} \frac{\sqrt{2ne\mathcal{B}}}{E_n + m} I_{n-1}(\tilde{\xi}) \\ -\frac{p_z}{E_n + m} I_n(\tilde{\xi}) \\ 0 \\ I_n(\tilde{\xi}) \end{pmatrix}. \quad (2.26)$$

where $\tilde{\xi}$ is obtained from ξ by changing the sign of the p_x -term.

For future use, we note down a few identities involving the spinors which can be obtained by direct substitutions of the solutions obtained above. The details of the calculation has been worked out in appendix A. The spin sum for the U -spinors is

$$\begin{aligned} P_U(y, y_\star, n, \mathbf{p}_y) &\equiv \sum_s U_s(y, n, \mathbf{p}_y) \bar{U}_s(y_\star, n, \mathbf{p}_y) \\ &= \frac{1}{2(E_n + m)} \times \left[\left\{ m(1 + \sigma_z) + \not{p}_\parallel - \tilde{\not{p}}_\parallel \gamma_5 \right\} I_{n-1}(\xi) I_{n-1}(\xi_\star) \right. \\ &\quad + \left\{ m(1 - \sigma_z) + \not{p}_\parallel + \tilde{\not{p}}_\parallel \gamma_5 \right\} I_n(\xi) I_n(\xi_\star) \\ &\quad - \sqrt{2ne\mathcal{B}} (\gamma_1 - i\gamma_2) I_n(\xi) I_{n-1}(\xi_\star) \\ &\quad \left. - \sqrt{2ne\mathcal{B}} (\gamma_1 + i\gamma_2) I_{n-1}(\xi) I_n(\xi_\star) \right], \end{aligned} \quad (2.27)$$

where the above notations for the 4-vectors and the \parallel and \perp notations has been defined in chapter 1, and $\sigma_z = i\gamma^1\gamma^2$. The symbol $\tilde{\not{p}}_\parallel = p^0\gamma_3 + p^3\gamma_0$. Similarly, the spin sum for the V -spinors can also be calculated, and we obtain

$$\begin{aligned} P_V(y, y_\star, n, \mathbf{p}_y) &\equiv \sum_s V_s(y, n, \mathbf{p}_y) \bar{V}_s(y_\star, n, \mathbf{p}_y) \\ &= \frac{1}{2(E_n + m)} \times \left[\left\{ -m(1 + \sigma_z) + \not{p}_\parallel - \tilde{\not{p}}_\parallel \gamma_5 \right\} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) \right. \\ &\quad + \left\{ -m(1 - \sigma_z) + \not{p}_\parallel + \tilde{\not{p}}_\parallel \gamma_5 \right\} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) \\ &\quad + \sqrt{2ne\mathcal{B}} (\gamma_1 - i\gamma_2) I_n(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) \\ &\quad \left. + \sqrt{2ne\mathcal{B}} (\gamma_1 + i\gamma_2) I_{n-1}(\tilde{\xi}) I_n(\tilde{\xi}_\star) \right]. \end{aligned} \quad (2.28)$$

2.3 The fermion field operator

Since we have found the solutions to the Dirac equation, we can now use them to construct the fermion field operator in the second quantized version. For this, we write

$$\psi(X) = \sum_{s=\pm} \sum_{n=0}^{\infty} \int \frac{dp_x dp_z}{D} [f_s(n, \mathbf{p}_y) e^{-ip \cdot X_y} U_s(y, n, \mathbf{p}_y) + \hat{f}_s^\dagger(n, \mathbf{p}_y) e^{ip \cdot X_y} V_s(y, n, \mathbf{p}_y)] \quad (2.29)$$

Here, $f_s(n, \mathbf{p}_y)$ is the annihilation operator for the fermion, and $\hat{f}_s^\dagger(n, \mathbf{p}_y)$ is the creation operator for the antifermion in the n -th Landau level with given values of p_x and p_z . The creation and annihilation operators satisfy the anticommutation relations

$$[f_s(n, \mathbf{p}_y), f_{s'}^\dagger(n', \mathbf{p}'_y)]_+ = \delta_{ss'} \delta_{nn'} \delta(p_x - p'_x) \delta(p_z - p'_z), \quad (2.30)$$

and a similar one with the operators \hat{f} and \hat{f}^\dagger , all other anticommutators being zero. The quantity D appearing in Eq. (2.29) depends on the normalization of the spinor solutions, and this is why the normalization of the spinors could have been chosen arbitrarily, as remarked after Eq. (2.18). Once we have chosen the spinor normalization, the factor D appearing in Eq. (2.29) is however fixed, and it can be determined from the equal time anticommutation relation

$$[\psi(X), \psi^\dagger(X_\star)]_+ = \delta^3(\mathbf{X} - \mathbf{X}_\star). \quad (2.31)$$

Plugging in the expression given in Eq. (2.29) to the left side of this equation and using the anticommutation relations of Eq. (2.30), we obtain

$$\begin{aligned} [\psi(X), \psi^\dagger(X_\star)]_+ &= \sum_s \sum_n \int \frac{dp_x dp_z}{D^2} \left(e^{-ip_x(x-x_\star)} e^{-ip_z(z-z_\star)} U_s(y, n, \mathbf{p}_y) U_s^\dagger(y_\star, n, \mathbf{p}_y) \right. \\ &\quad \left. + e^{ip_x(x-x_\star)} e^{ip_z(z-z_\star)} V_s(y, n, \mathbf{p}_y) V_s^\dagger(y_\star, n, \mathbf{p}_y) \right). \end{aligned} \quad (2.32)$$

Changing the signs of the dummy integration variables p_x and p_z in the second term, we can rewrite it as

$$\begin{aligned} [\psi(X), \psi^\dagger(X_\star)]_+ &= \sum_s \sum_n \int \frac{dp_x dp_z}{D^2} e^{-ip_x(x-x_\star)} e^{-ip_z(z-z_\star)} \left(U_s(y, n, \mathbf{p}_y) U_s^\dagger(y_\star, n, \mathbf{p}_y) \right. \\ &\quad \left. + V_s(y, n, -\mathbf{p}_y) V_s^\dagger(y_\star, n, -\mathbf{p}_y) \right). \end{aligned} \quad (2.33)$$

Using now the solutions for the U and the V spinors from Eqs. (2.24) and (2.26), it is straight forward to verify that

$$\begin{aligned} & \sum_s \left(U_s(y, n, \mathbf{p}_y) U_s^\dagger(y_\star, n, \mathbf{p}_y) + V_s(y, n, -\mathbf{p}_y) V_s^\dagger(y_\star, n, -\mathbf{p}_y) \right) \\ &= \left(1 + \frac{p_z^2 + 2ne\mathcal{B}}{(E_n + m)^2} \right) \times \text{diag} \left[I_{n-1}(\xi) I_{n-1}(\xi_\star), I_n(\xi) I_n(\xi_\star), I_{n-1}(\xi) I_{n-1}(\xi_\star), I_n(\xi) I_n(\xi_\star) \right], \end{aligned} \quad (2.34)$$

where ‘diag’ indicates a diagonal matrix with the specified entries, and ξ and ξ_\star involve the same value of p_x . At this stage, we can perform the sum over n in Eq. (2.33) using the completeness relation of Eq. (2.19), which gives the δ -function of the y -coordinate that should appear in the anticommutator. Finally, performing the integrations over p_x and p_z , we can recover the δ -functions for the other two coordinates as well, provided

$$\frac{2E_n}{E_n + m} \frac{1}{D^2} = \frac{1}{(2\pi)^2}, \quad (2.35)$$

using the expression for the energy eigenvalues from Eq. (2.20) to rewrite the prefactor appearing on the right side of Eq. (2.34). Putting the solution for D , we can rewrite Eq. (2.29) as

$$\begin{aligned} \psi(X) = & \sum_{s=\pm} \sum_{n=0}^{\infty} \int \frac{dp_x dp_z}{2\pi} \sqrt{\frac{E_n + m}{2E_n}} \\ & \times \left[f_s(n, \mathbf{p}_y) e^{-ip \cdot X_y} U_s(y, n, \mathbf{p}_y) + \hat{f}_s^\dagger(n, \mathbf{p}_y) e^{ip \cdot X_y} V_s(y, n, \mathbf{p}_y) \right]. \end{aligned} \quad (2.36)$$

The one-fermion states are defined as

$$|n, \mathbf{p}_y\rangle = C f_s^\dagger(n, \mathbf{p}_y) |0\rangle. \quad (2.37)$$

The normalization constant C is determined by the condition that the one-particle states should be orthonormal. For this, we need to define the theory in a finite but large region whose dimensions are L_x , L_y and L_z along the three spatial axes. This gives

$$C = \frac{2\pi}{\sqrt{L_x L_z}}. \quad (2.38)$$

Then

$$\psi_U(X) |n, \mathbf{p}_y\rangle = \sqrt{\frac{E_n + m}{2E_n L_x L_z}} e^{-ip \cdot X_y} U_s(y, n, \mathbf{p}_y) |0\rangle, \quad (2.39)$$

where ψ_U denotes the term in Eq. (2.36) that contains the U -spinors. Similarly,

$$\langle n, \mathbf{p}_y | \overline{\psi}_U(X) = \sqrt{\frac{E_n + m}{2E_n L_x L_z}} e^{ip \cdot X_y} \overline{U}_s(y, n, \mathbf{p}_y) \langle 0 |. \quad (2.40)$$

2.4 Inverse beta-decay

2.4.1 Preliminary comments

Now we have all the tools required for calculating the cross-section of the inverse beta-decay process in a magnetic field. We consider the possibility that the neutrons may be totally or partially polarized in the magnetic field, and find the cross-section as a function of this polarization. The neutrinos are assumed to be strictly standard model neutrinos, without any mass and consequent properties. The presence of the magnetic field breaks the isotropy of the background, and a careful calculation in this background reveals a dependence of the cross-section on the incident neutrino direction with respect to the magnetic field.

Considerable work has been done on the magnetic field dependence of the URCA processes which have neutrinos in their final states [17, 14, 15, 44, 45, 18]. An angular dependence obtained in the differential cross-section of these reactions imply that in a star with high magnetic field, neutrinos are created asymmetrically with respect to the magnetic field direction. The process that we consider, on the other hand, have neutrinos in the initial state. So this process influences the neutrino opacity in a star.

Some calculations of this process exist in the literature. Roulet [46], as well as Lai and Qian [47] performed the calculation by assuming that the magnetic field effects enter only through the phase space integrals, whereas the matrix element remains unaffected. Gvozdev and Ognev [44] considered the final electron to be exclusively in the lowest Landau level. Arras and Lai [45] calculated only the angular asymmetry, and only to the first order in the background magnetic field. Some earlier calculations [37, 48] did not take neutron polarization into account.

In this section the calculation of the cross-section for the inverse beta-decay process $\nu_e + n \rightarrow p + e^-$ in a background magnetic field is presented in full detail. The calculations involve evaluating the matrix element using spinor solutions of the electron in a magnetic field, taking all possible final Landau levels into account, including the possibility of neutron polarization, and performing the calculations to all orders in the background field in the 4-fermi interaction theory. The magnetic field might provide a net polarization of the neutrons, which is taken into account. However, the magnitude of the field is assumed to be much smaller than m_p^2/e , so its effects on the proton spinors are ignored. The electron spinors, on the other hand, are the ones appropriate for the Landau levels. Thus, we can

write the process as

$$\nu_e(q) + n(P) \rightarrow p(P') + e(\mathbf{p}'_e, n') . \quad (2.41)$$

2.4.2 The S -matrix element

The charged current interaction Lagrangian for this process is given by Eq. (1.3). Using it in first order perturbation, the S -matrix element between the final and the initial states of the process in Eq. (2.41) is given by

$$S_{fi} = \sqrt{2} G_\beta \int d^4 X \left\langle e(\mathbf{p}'_e, n') \left| \bar{\psi}_{(e)} \gamma^\mu L \psi_{(\nu_e)} \right| \nu_e(q) \right\rangle \\ \times \left\langle p(P') \left| \bar{\psi}_{(p)} \gamma_\mu (G_V - G_A \gamma_5) \psi_{(n)} \right| n(P) \right\rangle . \quad (2.42)$$

For the hadronic part, we should use the usual solutions of the Dirac field which are normalized within a box of volume V , and this gives

$$\left\langle p(P') \left| \bar{\psi}_{(p)} \gamma_\mu (G_V - G_A \gamma_5) \psi_{(n)} \right| n(P) \right\rangle = \frac{e^{i(P'-P)\cdot X}}{\sqrt{2\mathcal{E}V}\sqrt{2\mathcal{E}'V}} \left[\bar{u}_{(p)}(\mathbf{P}') \gamma_\mu (G_V - G_A \gamma_5) u_{(n)}(\mathbf{P}) \right] , \quad (2.43)$$

using the notations $\mathcal{E} = P_0$ and $\mathcal{E}' = P'_0$. For the leptonic part, we need to take into account the magnetic spinors for the electron. Using Eq. (2.40), we obtain

$$\left\langle e(\mathbf{p}'_e, n') \left| \bar{\psi}_{(e)} \gamma^\mu L \psi_{(\nu_e)} \right| \nu_e(q) \right\rangle = \frac{e^{-iq\cdot X + ip'\cdot X_e}}{\sqrt{2\Omega V}} \sqrt{\frac{E_{n'} + m}{2E_{n'}L_xL_z}} \left[\bar{U}_{(e)}(y, n', \mathbf{p}'_e) \gamma^\mu L u_{(\nu_e)}(\mathbf{q}) \right] . \quad (2.44)$$

Here $\Omega = q_0$. The symbol m was used previously to denote the mass of an arbitrary charged particle, from now onwards it will denote the mass of an electron in this chapter. Putting these back into Eq. (2.42) and performing the integrations over all co-ordinates except y , we obtain

$$S_{fi} = (2\pi)^3 \delta^3_y(P + q - P' - p') \left[\frac{E_{n'} + m}{2\Omega V 2\mathcal{E}V 2\mathcal{E}'V 2E_{n'}L_xL_z} \right]^{1/2} \mathcal{M}_{fi} . \quad (2.45)$$

Here, δ^3_y implies, in accordance with the notation introduced earlier, the δ -function for all space-time co-ordinates except y . Contrary to the field-free case, we do not get 4-momentum conservation because the y -component of momentum is not a good quantum number in this problem. The quantity \mathcal{M}_{fi} is the Feynman amplitude, given by

$$\mathcal{M}_{fi} = \sqrt{2} G_\beta \left[\bar{u}_{(p)}(\mathbf{P}') \gamma_\mu (G_V - G_A \gamma_5) u_{(n)}(\mathbf{P}) \right] \int dy e^{iy\cdot y} \left[\bar{U}_{(e)}(y, n', \mathbf{p}'_e) \gamma^\mu L u_{(\nu_e)}(\mathbf{q}) \right] , \quad (2.46)$$

using the shorthand

$$u_y = P_y + q_y - P'_y. \quad (2.47)$$

The transition rate in a large time T is given by $|S_{fi}|^2/T$. From Eq. (2.45), using the usual rules like

$$\begin{aligned} \left| \delta(\mathcal{E} + \Omega - \mathcal{E}' - E_{n'}) \right|^2 &= \frac{T}{2\pi} \delta(\mathcal{E} + \Omega - \mathcal{E}' - E_{n'}), \\ \left| \delta(P_x + q_x - P'_x - p'_x) \right|^2 &= \frac{L_x}{2\pi} \delta(P_x + q_x - P'_x - p'_x), \\ \left| \delta(P_z + q_z - P'_z - p'_z) \right|^2 &= \frac{L_z}{2\pi} \delta(P_z + q_z - P'_z - p'_z), \end{aligned} \quad (2.48)$$

we obtain

$$|S_{fi}|^2/T = \frac{1}{16}(2\pi)^3 \delta_y^3(P + q - P' - p') \frac{E_{n'} + m}{V^3 \Omega \mathcal{E} \mathcal{E}' E_{n'}} |\mathcal{M}_{fi}|^2. \quad (2.49)$$

2.4.3 The scattering cross-section

Using unit flux $1/V$ for the incident particle as usual, we can write the differential cross-section as

$$d\sigma = V \frac{|S_{fi}|^2}{T} d\rho, \quad (2.50)$$

where $d\rho$, the differential phase space for final particles, is given in our case by

$$d\rho = \frac{L_x}{2\pi} dp'_x \frac{L_z}{2\pi} dp'_z \frac{V}{(2\pi)^3} d^3 P'. \quad (2.51)$$

Therefore

$$\begin{aligned} d\sigma &= V \frac{|S_{fi}|^2}{T} \frac{L_x L_z}{(2\pi)^2} dp'_x dp'_z \frac{V}{(2\pi)^3} d^3 P' \\ &= \frac{1}{64\pi^2} \delta_y^3(P + q - P' - p') \frac{E_{n'} + m}{\Omega \mathcal{E} \mathcal{E}' E_{n'}} |\mathcal{M}_{fi}|^2 \frac{L_x L_z}{V} dp'_x dp'_z d^3 P'. \end{aligned} \quad (2.52)$$

The square of the matrix element is

$$|\mathcal{M}_{fi}|^2 = 2G_\beta^2 \ell^{\mu\nu} H_{\mu\nu}, \quad (2.53)$$

where $H_{\mu\nu}$ is the hadronic part and $\ell^{\mu\nu}$ the leptonic part, whose calculation we outline now.

For the hadronic part, we can use the usual Dirac spinors because of our assumption that the magnetic field is much smaller than m_p^2/e . We will work in the rest frame of

the neutron. Due to the presence of the background magnetic field, the neutrons may be totally or partially polarized. We define the quantity

$$S \equiv \frac{N_n^{(+)} - N_n^{(-)}}{N_n^{(+)} + N_n^{(-)}}, \quad (2.54)$$

where $N_n^{(\pm)}$ denote the number of neutrons parallel and antiparallel to the magnetic field. Then

$$H_{\mu\nu} = \frac{1}{2}(1+S)H_{\mu\nu}^{(+)} + \frac{1}{2}(1-S)H_{\mu\nu}^{(-)}, \quad (2.55)$$

where $H_{\mu\nu}^{(\pm)}$ denotes the contribution calculated with spin-up and spin-down neutrons respectively. Either of these contributions can be calculated by using the spin projection operator, which is $\frac{1}{2}(1 \pm \gamma_5 \gamma_3)$ for up and down spins. A straight forward calculation then yields

$$\begin{aligned} H_{\mu\nu} = & 2(G_V^2 + G_A^2)(P_\mu P'_\nu + P_\nu P'_\mu - g_{\mu\nu} P \cdot P') \\ & + 2(G_V^2 - G_A^2)m_n m_p g_{\mu\nu} + 4iG_V G_A \varepsilon_{\mu\nu\lambda\rho} P^\lambda P'^\rho \\ & - S[4G_V G_A m_n (P'_\mu g_{3\nu} + P'_\nu g_{3\mu} - P'_3 g_{\mu\nu}) + 2i\varepsilon_{\mu\nu 3\alpha} R^\alpha], \end{aligned} \quad (2.56)$$

where we have introduced the shorthand

$$R^\alpha = (G_V^2 + G_A^2)m_n P'^\alpha - (G_V^2 - G_A^2)m_p P^\alpha. \quad (2.57)$$

We have omitted some terms in the expression for $H_{\mu\nu}$ that involve spatial components of the neutron momentum, with the anticipation that we will perform the calculation in the neutron rest frame.

In the leptonic part $\ell^{\mu\nu}$, we should use the magnetic spinors given in section 2.2. This gives

$$\ell^{\mu\nu} = \int dy \int dy_* e^{iu_y(y_* - y)} \text{Tr}[P_U(y, y_*, n', \mathbf{p}_y') \gamma^\mu \not{q} \gamma^\nu L], \quad (2.58)$$

where P_U denotes the spinor sum for the electrons, given in Eq. (2.27). We now have to perform the integrations over y and y_* . Each of these variables should be integrated in the range $-\frac{1}{2}L_y$ to $+\frac{1}{2}L_y$. However, since we will take the infinite volume limit at the end as usual, we let $L_y \rightarrow \infty$ and use the result [49]

$$\int_{-\infty}^{+\infty} dy e^{ixy} I_n(y) = i^n \sqrt{2\pi} I_n(x). \quad (2.59)$$

This gives

$$\ell^{\mu\nu} = \frac{2\pi}{e\mathcal{B}} \frac{1}{(E_{n'} + m)} (\Lambda^\mu q^\nu + \Lambda^\nu q^\mu - q \cdot \Lambda g^{\mu\nu} - i\varepsilon^{\mu\nu\alpha\beta} \Lambda_\alpha q_\beta), \quad (2.60)$$

where

$$\begin{aligned} \Lambda^\alpha &= \left[I_{n'-1} \left(\frac{u_y}{\sqrt{e\mathcal{B}}} \right) \right]^2 (p'_\parallel^\alpha - \tilde{p}'_\parallel^\alpha) + \left[I_{n'} \left(\frac{u_y}{\sqrt{e\mathcal{B}}} \right) \right]^2 (p'_\parallel^\alpha + \tilde{p}'_\parallel^\alpha) \\ &\quad - 2\sqrt{2n'e\mathcal{B}} g_2^\alpha I_{n'} \left(\frac{u_y}{\sqrt{e\mathcal{B}}} \right) I_{n'-1} \left(\frac{u_y}{\sqrt{e\mathcal{B}}} \right). \end{aligned} \quad (2.61)$$

Thus,

$$\begin{aligned} |\mathcal{M}_{fi}|^2 &= 8G_\beta^2 \times \frac{2\pi}{e\mathcal{B}} \frac{1}{(E_{n'} + m)} \left[(G_V^2 + G_A^2)(P \cdot \Lambda P' \cdot q + P' \cdot \Lambda P \cdot q) \right. \\ &\quad - (G_V^2 - G_A^2)m_n m_p q \cdot \Lambda - 2G_V G_A(P \cdot \Lambda P' \cdot q - P' \cdot \Lambda P \cdot q) \\ &\quad \left. + S(2G_V G_A m_n (P' \cdot \Lambda q_z + P' \cdot q \Lambda_z) - \Lambda_z q \cdot R + q_z \Lambda \cdot R) \right]. \end{aligned} \quad (2.62)$$

We now choose the axes such that the 3-momentum of the incoming neutrino is in the x - z plane. We will also assume that $|\mathbf{P}'| \ll m_p$ for the range of energies of interest to us. In that case, it is easy to see that the terms involving $\sqrt{2n'e\mathcal{B}}$ drop out, and we obtain

$$\begin{aligned} |\mathcal{M}_{fi}|^2 &= 8G_\beta^2 \times \frac{2\pi}{e\mathcal{B}} \frac{m_n m_p}{E_{n'} + m} \times \left[(G_V^2 + 3G_A^2)\Omega\Lambda_0 + (G_V^2 - G_A^2)q_z\Lambda_z \right. \\ &\quad \left. + 2G_A S((G_V - G_A)\Omega\Lambda_z + (G_V + G_A)q_z\Lambda_0) \right]. \end{aligned} \quad (2.63)$$

We now put this expression into Eq. (2.52) and calculate the total cross-section by performing the integrations over different final state momenta appearing in that formula. First we integrate over P'_x and P'_z . These appear only in the momentum conserving δ -function. Integration over them therefore just gets rid of the corresponding δ -functions. For the integration over p'_x , we refer to Eq. (2.13). Since the center of the oscillator has to lie between $-\frac{1}{2}L_y$ and $\frac{1}{2}L_y$, we conclude that $-\frac{1}{2}L_y e\mathcal{B} \leq p'_x \leq \frac{1}{2}L_y e\mathcal{B}$. Thus the integration over p'_x gives a factor $L_y e\mathcal{B}$.

Putting back into Eq. (2.52) and using $V = L_x L_y L_z$, we obtain

$$\begin{aligned} d\sigma &= \frac{G_\beta^2}{4\pi} \frac{\delta(D + \Omega - E_{n'})}{\Omega E_{n'}} \times \left[(G_V^2 + 3G_A^2)\Omega\Lambda_0 + (G_V^2 - G_A^2)q_z\Lambda_z \right. \\ &\quad \left. + 2G_A S((G_V - G_A)\Omega\Lambda_z + (G_V + G_A)q_z\Lambda_0) \right] dP'_y dP'_z, \end{aligned} \quad (2.64)$$

where D is the neutron-proton mass difference, $m_n - m_p$.

We next perform the integration over P'_y . In the integrand, it occurs only as the argument of the functions I_n and I_{n-1} . The functions I_n are orthogonal in the sense that

$$\int_{-\infty}^{+\infty} da I_n(a) I_{n'}(a) = \sqrt{e\mathcal{B}} \delta_{nn'} . \quad (2.65)$$

This property can be used to perform the integration over P'_y . We have already remarked that the term proportional to $\sqrt{2n'e\mathcal{B}}$ in Eq. (2.61) does not contribute. From other two terms, we obtain

$$\begin{aligned} \int dP'_y \Lambda^\alpha &= e\mathcal{B} [(p'_\parallel - \tilde{p}'_\parallel)(1 - \delta_{n',0}) + (p'_\parallel + \tilde{p}'_\parallel)] \\ &= e\mathcal{B} [g_{n'} p'_\parallel + \delta_{n',0} \tilde{p}'_\parallel] , \end{aligned} \quad (2.66)$$

where

$$g_{n'} = 2 - \delta_{n',0} \quad (2.67)$$

gives the degeneracy of the Landau level. Notice the appearance of the Kronecker delta, $\delta_{n',0}$, in the expression of Eq. (2.66). The reason for this is that, while two terms of Eq. (2.61) contribute in the integral for $n' \neq 0$, only one of them contributes for $n' = 0$ since $I_{-1} = 0$.

The final integration is over p'_z . Writing the argument of the remaining δ -function in terms of p'_z , we find that the zeros occur when

$$p'_z = p'_\pm \equiv \pm \sqrt{(D + \Omega)^2 - m^2 - 2n'e\mathcal{B}} . \quad (2.68)$$

Therefore,

$$\delta(D + \Omega - E_{n'}) = \frac{D + \Omega}{\sqrt{(D + \Omega)^2 - m^2 - 2n'e\mathcal{B}}} (\delta(p'_z - p'_+) + \delta(p'_z - p'_-)) . \quad (2.69)$$

In the integration, the terms proportional to p'_z in the integrand receive equal and opposite contributions from the two δ functions and cancel. For the other terms, independent of p'_z , both the contributions are equal. So we obtain

$$\begin{aligned} \sigma_{n'} &= \frac{e\mathcal{B}G_\beta^2}{2\pi} \left[g_{n'} \left\{ (G_V^2 + 3G_A^2) + 2G_AS(G_V + G_A) \cos\theta \right\} \right. \\ &\quad \left. + \delta_{n',0} \left\{ (G_V^2 - G_A^2) \cos\theta + 2G_AS(G_V - G_A) \right\} \right] \frac{D + \Omega}{\sqrt{(D + \Omega)^2 - m^2 - 2n'e\mathcal{B}}} , \end{aligned} \quad (2.70)$$

where we have defined the direction of the incoming neutrino by the angle θ , with

$$q_z = \Omega \cos \theta. \quad (2.71)$$

In Eq. (2.70), we have denoted the cross-section by $\sigma_{n'}$ because the electron ends up in a specific Landau level n' . The total cross-section is then given as a sum over all possible values of n' , i.e.,

$$\begin{aligned} \sigma = \sum_{n'=0}^{n'_{\max}} \sigma_{n'} &= \frac{e\mathcal{B}G_\beta^2}{2\pi} \sum_{n'=0}^{n'_{\max}} \left[g_{n'} \left\{ (G_V^2 + 3G_A^2) + 2G_A S(G_V + G_A) \cos \theta \right\} \right. \\ &\quad \left. + \delta_{n',0} \left\{ (G_V^2 - G_A^2) \cos \theta + 2G_A S(G_V - G_A) \right\} \right] \frac{D + \Omega}{\sqrt{(D + \Omega)^2 - m^2 - 2n'\mathcal{B}}}. \end{aligned} \quad (2.72)$$

The possible allowed Landau level has a maximum, n'_{\max} , which is given by the fact that the quantity under the square root sign in the denominator of Eq. (2.72) must be non-negative, i.e.,

$$n'_{\max} = \text{int} \left\{ \frac{1}{2e\mathcal{B}} [(D + \Omega)^2 - m^2] \right\}. \quad (2.73)$$

Eq. (2.72) gives our result for the cross-section of the inverse beta decay process. Some properties of this formula are worth noting.

For unpolarized neutrons, $S = 0$, the cross-section for $n' \neq 0$ does not depend on the direction of the incoming neutrino. The same is not true if the electron ends up in the lowest Landau level. The cross-section will be asymmetric in this case.

All terms in the cross-section which depend on S have a common factor G_A . The reason is that, if G_A were equal to zero, the interaction in the hadronic sector would have been spin-independent.

If the final electron is in the lowest Landau level and the initial neutrino momentum is antiparallel to the magnetic field, Eq. (2.72) shows that

$$\sigma_0 = \frac{e\mathcal{B}G_\beta^2}{2\pi} \left[4G_A^2(1 - S) \right] \frac{D + \Omega}{\sqrt{(D + \Omega)^2 - m^2}}. \quad (2.74)$$

Note that the vector coupling G_V does not contribute to the cross-section in this limit. This can be understood easily. The neutrino spin is along the $+z$ direction whereas the electron spin in the lowest Landau level must be in the $-z$ direction. Thus there is a spin-flip in the leptonic sector. Conservation of angular momentum then implies that

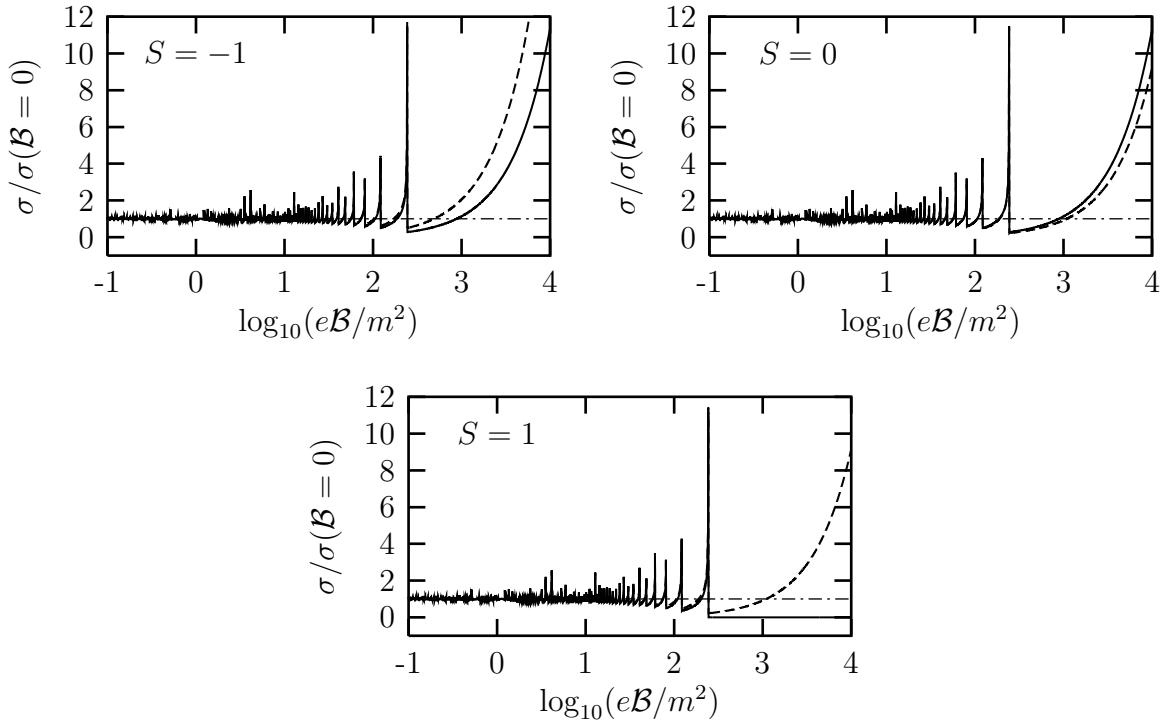


Figure 2.1: Enhancement of cross-section in a magnetic field for an initial neutrino energy of 10 MeV. Different panels show the results for different net polarizations of the neutrons. The solid and the dashed lines correspond to the initial neutrino momentum antiparallel and parallel to the magnetic field. Each curve has been normalized by the cross-section in field-free case for the same values of S and $\cos \theta$. The horizontal dashed lines represent unity in the vertical scale.

there must be a spin-flip in the hadronic sector as well. In the non-relativistic limit for hadrons that we have employed, this can occur only through the axial coupling.

If further we consider totally polarized neutrons, i.e., $S = 1$, we see that σ_0 vanishes. Again, this is a direct consequence of angular momentum conservation. Since both initial particles have spin up, angular momentum conservation requires both final particles in spin up states as well. But the spin-up state is not available for the electron in the lowest Landau level.

It is instructive to check that the result obtained in Eq. (2.72) reduces to the known result for the field-free case. The contribution specific to the zeroth Landau level vanishes in the limit $\mathcal{B} \rightarrow 0$ owing to the overall factor of $e\mathcal{B}$. The other terms also have the factor

$e\mathcal{B}$, but in this case we also need to sum over infinitely many states. This gives

$$\sigma = \frac{e\mathcal{B}G_\beta^2}{\pi} \left[(G_V^2 + 3G_A^2) + 2G_AS(G_V + G_A)\cos\theta \right] \times \left(\sum_{n'=0}^{n'_{\max}} \frac{D + \Omega}{\sqrt{(D + \Omega)^2 - m^2 - 2n'e\mathcal{B}}} - \frac{D + \Omega}{2\sqrt{(D + \Omega)^2 - m^2}} \right). \quad (2.75)$$

For $\mathcal{B} \rightarrow 0$, the last term vanishes, and we can identify n'_{\max} as the integer for which the denominator of the summand vanishes. Thus we obtain

$$\begin{aligned} \sigma &\longrightarrow \frac{e\mathcal{B}G_\beta^2}{\pi} \left[(G_V^2 + 3G_A^2) + 2G_AS(G_V + G_A)\cos\theta \right] \int_0^{n'_{\max}} dn' \frac{D + \Omega}{\sqrt{(D + \Omega)^2 - m^2 - 2n'e\mathcal{B}}} \\ &= \frac{G_\beta^2}{\pi} \left[(G_V^2 + 3G_A^2) + 2G_AS(G_V + G_A)\cos\theta \right] (D + \Omega) \sqrt{(D + \Omega)^2 - m^2}, \end{aligned} \quad (2.76)$$

which is the correct result in the field-free case.

In Fig. 2.1, we have plotted the ratio of the cross-section to its corresponding value at $\mathcal{B} = 0$ as a function of the magnetic field. The plots have been done for unpolarized ($S = 0$) as well as totally polarized neutrons along ($S = 1$) and opposite ($S = -1$) to the magnetic field, with the initial neutrino momentum parallel and antiparallel to the magnetic field. For $S = -1$, we find that neutrinos parallel to the magnetic field have a smaller cross-section than those antiparallel to the field, and the difference is pronounced for large fields. For $S = 0$, the situation is just reversed. For $S = 1$, if the magnetic field is high enough so that $n'_{\max} = 0$, we see that the cross-section vanishes for neutrino momentum antiparallel to the field. The reason for this has already been discussed.

For the inverse beta decay process, which is symbolically written as in Eq. (2.41), the cross-section has been calculated earlier by several authors [46, 47]. They assumed that the matrix element remains unaffected by the magnetic field, only the modified phase space integral makes the difference in the cross-section. The magnetic field effects enters only through the following modification of the phase space integral and the spin factor of the electron:

$$2 \int \frac{d^3 p}{(2\pi)^3} \longrightarrow \frac{e\mathcal{B}}{(2\pi)^2} \sum_{n'} g_{n'} \int dp_z, \quad (2.77)$$

where $g_{n'}$ is the degeneracy of the n' -th Landau level as explained in Eq. (2.67). The sum over n' is restricted to the region

$$n' < \frac{(D + \Omega)^2 - m_e^2}{2e\mathcal{B}}, \quad (2.78)$$

where Ω is the neutrino energy. The results they obtained is the same as the term proportional to $G_V^2 + 3G_A^2$ that we obtained.

Subsequent calculations incorporated the modification of wave functions. Arras and Lai [45], while still treating the nucleons as non-relativistic, used the non-relativistic Landau levels as well as the finiteness of the recoil energy for the proton. They found the cross-section and went on to derive expressions for the neutrino opacity. From the final expressions, one can only recognize the terms linear in \mathcal{B} . The opacity was calculated also by Chandra, Goyal and Goswami [50]. Like the previous authors, they also considered the contribution to the opacity from other reactions like neutrino-nucleon elastic scattering.

2.4.4 Consequences of neutrino energy spread

The enhancement factor in Fig. 2.1 shows some spikes. They appear at values of the magnetic field for which the denominator of Eq. (2.72) vanishes for some n' . For field values larger than this, that particular Landau level does not contribute to the cross-section. To the right of the final spike that appears in the figure, only the zeroth Landau level contributes. In other words, the final electron can go only to the lowest Landau level for such high values of the magnetic field. The exact value of \mathcal{B} for which this occurs depends of course on the energy of the initial neutrino.

We need to make an important point about these spikes. Each spike in fact go all the way up to infinity. The finite height of a spike in the figure is an artifact of the finite step size taken in plotting it.

In reality, of course, a cross-section cannot be infinite. In the present case, the cross-section remains finite due to the fact that the initial particles cannot be exactly monochromatic due to the uncertainty relation. For an example if we concentrate on the initial neutrino alone then there must be a spread in its energy, which can be represented by a probability distribution $\Phi(\Omega)$, defined by

$$\int d\Omega \Phi(\Omega) = 1. \quad (2.79)$$

In that case, the cross-section in a real experiment should be written in the form

$$\sigma = \int d\Omega \Phi(\Omega) \sigma(\Omega), \quad (2.80)$$

where $\sigma(\Omega)$ is the expression derived in Eq. (2.72) for a single value of energy.

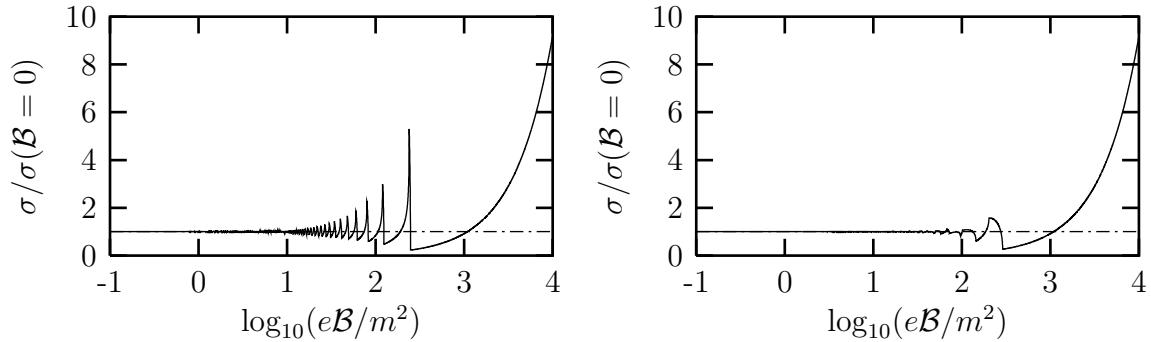


Figure 2.2: Cross-section for unpolarized neutrons as a function of the magnetic field for a flat energy distribution, normalized to the cross-section in the field-free case. The initial neutrino momentum is along the magnetic field, and energy is 10 MeV. The energy spread $\Omega_2 - \Omega_1$ is 0.2 MeV for the left panel and 2 MeV for the right panel.

As an illustration, we consider the case of unpolarized neutrons ($S = 0$), and take a flat probability distribution of initial neutrino energy, viz.,

$$\Phi(\Omega) = \begin{cases} \frac{1}{\Omega_2 - \Omega_1} & \text{if } \Omega_1 \leq \Omega \leq \Omega_2, \\ 0 & \text{otherwise.} \end{cases} \quad (2.81)$$

Then the integration of Eq. (2.80) gives

$$\sigma = \frac{e\mathcal{B}G_\beta^2}{2\pi(\Omega_2 - \Omega_1)} [F(\Omega_2) - F(\Omega_1)], \quad (2.82)$$

where

$$F(\Omega) = \sum_{n'=0}^{n'_{\max}} \left[g_{n'}(G_V^2 + 3G_A^2) + \delta_{n',0}(G_V^2 - G_A^2) \cos \theta \right] \times \sqrt{(D + \Omega)^2 - m^2 - 2n'e\mathcal{B}}, \quad (2.83)$$

with n'_{\max} determined by Eq. (2.73). Fig. 2.2 shows the variation of this quantity with the magnetic field for $\cos \theta = 1$. In this figure, we normalize the cross-section by $\mathcal{B} = 0$ cross-section with the energy distribution of Eq. (2.81), which is

$$\sigma(\mathcal{B} = 0) = \frac{G_\beta^2(G_V^2 + 3G_A^2)}{3\pi(\Omega_2 - \Omega_1)} \left([(D + \Omega_2)^2 - m^2]^{3/2} - [(D + \Omega_1)^2 - m^2]^{3/2} \right). \quad (2.84)$$

Keeping the central value of neutrino energy as 10 MeV as before, we have drawn these plots for two different values of the spread, as mentioned in the caption. For the smaller

value of the spread in particular, the graph looks very similar to that drawn in Fig. 2.1, but the difference is that now the height of the spikes denote the actual enhancement, and is not an artifact of the plotting procedure. For the higher value of the energy spread, we see that the spikes have smoothed out.

2.5 Asymmetric emission of neutrinos from a proto-neutron star

From what has been discussed so far we know that the inverse beta decay cross-section is direction sensitive through its dependence on the angle θ , the angle between the neutrino 3-momentum and the magnetic field direction. This anisotropic effect in the cross-section can have interesting astrophysical applications in explaining the high velocities of pulsars, of the order of $450 \pm 90 \text{ Km s}^{-1}$. Typical pulsars have masses between $1.0M_{\odot}$ and $1.5M_{\odot}$, i.e., about $2 \times 10^{33}\text{g}$. The momentum associated with the proper motion of a pulsar would therefore be of order 10^{41} g cm/s . On the other hand the energy carried off by neutrinos in a supernova explosion is about $3 \times 10^{53}\text{erg}$, which corresponds to a sum of magnitudes of neutrino momenta of 10^{43}g cm/s . Thus an asymmetry of order of 1% in the distribution of the outgoing neutrinos would explain the kick of the pulsars. It has been argued that an asymmetry of this order in the distribution of outgoing neutrinos can be generated by the anisotropic cross-sections of the various neutrino related processes in presence of a constant magnetic field [16, 51, 52, 48, 43, 37]. A simple calculation can make the point clear.

The typical size of a neutron star is about $R \approx 10 \text{ km}$. Thus, when a neutrino, generated at the core reaches a density where its mean free path is about R , it escapes from the star. Therefore the condition for the neutrino to escape can be written as

$$n_n \sigma = \frac{1}{R}, \quad (2.85)$$

where n_n is the neutron number density. We already observed that σ is direction dependent. Therefore, the value of n_n on the “neutrino sphere” depends on the direction as well, and the surface is no longer a sphere. Different values of n_n will correspond to different temperatures. Thus, neutrinos will be emitted with different momenta in different directions. This can result in a kick to the star.

To estimate the magnitude of the kick, let us abbreviate Eq. (2.72) as

$$\sigma = e\mathcal{B}G_{\beta}^2(a + b \cos \theta). \quad (2.86)$$

If we now consider the directions $\theta = 0$ and $\theta = \pi$, the difference in neutron density on the corresponding points on the neutrino surface is given by

$$\Delta n_n = \frac{1}{e\mathcal{B}G_\beta^2 R} \left(\frac{1}{a-b} - \frac{1}{a+b} \right) = \frac{2b}{e\mathcal{B}G_\beta^2 a^2 R}, \quad (2.87)$$

neglecting corrections of order b/a .

The neutron gas in a typical proto-neutron star can be considered to be non-relativistic and degenerate. The number density of neutrons is thus given by [53]

$$n_n = \frac{p_F^3}{3\pi^2} \left[1 + \frac{\pi^2 m_n^2 T^2}{2p_F^4} + \dots \right], \quad (2.88)$$

where p_F is the Fermi momentum, and we have neglected higher order terms in the temperature. This gives

$$\frac{dn_n}{dT} = \frac{m_n^2}{3} \left(\frac{T}{p_F} \right). \quad (2.89)$$

So the temperature difference between the points on the neutrino surface in the $\theta = 0$ and $\theta = \pi$ directions is

$$\Delta T = \frac{3}{m_n^2} \frac{p_F}{T} \frac{2b}{e\mathcal{B}G_\beta^2 a^2 R} \quad (2.90)$$

The momentum asymmetry can now be written as

$$\frac{\Delta|\mathbf{q}|}{|\mathbf{q}|} = \frac{1}{6} \cdot \frac{4\Delta T}{T}, \quad (2.91)$$

where we have assumed a black body radiation luminosity ($\propto T^4$) for the effective neutrino surface. The factor 1/6 comes in because the asymmetry pertains only to ν_e , whereas 6 types of neutrinos and antineutrinos contribute to the energy emitted. This gives

$$\frac{\Delta|\mathbf{q}|}{|\mathbf{q}|} = \frac{4}{m_n^2} \frac{p_F}{T^2} \frac{b}{e\mathcal{B}G_\beta^2 a^2 R}, \quad (2.92)$$

To find p_F , we use the leading term in Eq. (2.88) and estimate n_n from the equation

$$n_n = \frac{\rho(1-Y_e)}{m_p}, \quad (2.93)$$

where Y_e is the electron fraction and ρ is the mass density. Taking $Y_e = 1/10$, we obtain

$$p_F = 24\rho_{11}^{1/3} \text{ MeV}. \quad (2.94)$$

where ρ_{11} is the mass density in units of $10^{11} \text{ g cm}^{-3}$. Putting this back in Eq. (2.92), we obtain

$$\left| \frac{\Delta|\mathbf{q}|}{|\mathbf{q}|} \right| = 27\rho_{11}^{1/3} \mathcal{B}_{14}^{-1} T_{\text{MeV}}^{-2} \frac{|b|}{a^2}, \quad (2.95)$$

where $\mathcal{B}_{14} = \mathcal{B}/(10^{14} \text{ Gauss})$ and $T_{\text{MeV}} = T/(1 \text{ MeV})$. For $\Omega \gg m_e$ which is the relevant case and $S = 0$ as a typical value of neutron polarization we can find out the values of the constants a and b . The values of both of them will in general depend upon n'_{\max} but if we take only the contribution from the $n' = 0$ level then

$$a = \frac{G_V^2 + 3G_A^2}{2\pi} = 9.2 \times 10^{-1}. \quad (2.96)$$

and

$$b = \frac{G_V^2 - G_A^2}{2\pi} = -9.3 \times 10^{-2}, \quad (2.97)$$

using $G_V = 1$ and $G_A = -1.26$. This gives

$$\left| \frac{\Delta|\mathbf{q}|}{|\mathbf{q}|} \right| = 3\rho_{11}^{1/3} \mathcal{B}_{14}^{-1} T_{\text{MeV}}^{-2}. \quad (2.98)$$

Obviously, with reasonable choices of ρ , \mathcal{B} and T , it is possible to obtain a fractional momentum imbalance of the order of 1% which is necessary for explaining the pulsar kicks.

2.6 Conclusion

This chapter started with a discussion on neutrino-electron scattering and other processes related to it by crossing. The basic theory of spinor solutions in an external magnetic field has been exposed with some detail in next two sections. A formal field theoretical framework has been presented based on which cross-sections of any process, which contains charged particles in the initial or the final state, can be calculated. The main calculations concerning the inverse beta decay cross-section start from the fourth section.

The calculations show that for the inverse beta-decay process, even for unpolarized neutrons, the cross-section depends on the direction of the neutrino momentum. This asymmetry is not surprising since the background magnetic field makes it an anisotropic problem. A similar asymmetry has been noted for URCA processes [18], where the neutrino is in the final state. From Eq. (2.72) it is seen that the anisotropy in the cross-section

comes only from the zeroth Landau level contribution of the electron [44]. However, for the higher levels, there is a cancellation between the two possible states in a Landau level which washes out all angular dependence in these levels, provided the neutrons are unpolarized. The asymmetry in cross-section will therefore come only from the the zeroth Landau level state of the electrons and its amount will depend on the relative contribution of this state to the total cross-section. If the magnetic field is so high that only the zeroth Landau level state can be obtained for the electron, the asymmetry will be large, about 18%. For smaller and smaller magnetic fields, the asymmetry decreases with new Landau levels contributing. For polarized neutrons, however, there is an asymmetry even in the field-free case as is evident from Eq. (2.76). In presence of a magnetic field, the asymmetry will in general depend on the magnetic field, as it appears from various plots in Fig. 2.1.

From Eq. (2.76) it is seen that the inverse beta decay cross-section has a smooth $\mathcal{B} \rightarrow 0$ limit. From the curves in Fig. 2.1 it is observed that the cross-section has spikes, which go all the way to infinity, for specific values of the magnetic field magnitude. Subsection 2.4.4 deals with the issue of these infinite spikes. It is shown that if instead of a monochromatic neutrino the initial neutrino has an energy spread, as is expected in a real situation, then the spikes gets smoothed up. This fact is evident from Fig. 2.2.

The anisotropic nature of the inverse beta decay cross-section can have far reaching consequences for neutrino emission from a proto-neutron star as has been shown in the section concerning asymmetric emission of neutrinos. It has been discussed in the literature that the presence of asymmetric magnetic fields can cause asymmetric neutrino emission from a proto-neutron star [54]. However, the calculations in this chapter show that even with a uniform magnetic field, neutrino emission would be asymmetric because of the $\cos \theta$ -dependent terms in the cross-section.

CHAPTER 3

Observable effects due to intermediate virtual charged particles in a magnetic field

3.1 Introduction

In quantum field theoretic calculations, the spinors given in the previous chapter in Eq. (2.24) and Eq. (2.26) should be used if the charged particle appears in the initial or the final state of a physical process. If, on the other hand, the charged particle appears in the internal lines of the Feynman diagram of the process then we should use its propagator.

This chapter is on propagators for charged particles, mainly fermions, in presence of a background uniform magnetic field. The calculations will be extrapolated to the case where there is a background medium also. For the medium modification of the propagators a brief discussion on statistical field theory (commonly called finite temperature field theory) is discussed in the appendix C. The Schwinger propagator is introduced in this chapter and is followed by a discussion on the phase factor accompanying it. A detailed discussion on the derivation of the Schwinger propagator is supplied in [55]. The phase factor of the Schwinger propagator is dealt in some detail. In most of the works this phase factor is treated with irrelevance but the reason is rarely explained.

In the chapter 1 we noticed that the standard model neutrinos do not have any magnetic moment and so they cannot interact with external magnetic fields. In going beyond standard model i.e. by including right handed neutrinos we can generate a magnetic moment of the neutrinos. Magnetic fields can also affect the neutrino properties in a different way i.e., through the charged particle propagators as discussed in this chapter. This indirect effect of the magnetic fields from the propagators can affect neutrino self-energy calculations. Without doing actual loop calculations it is possible to predict the form of

the self-energy expressions in a magnetic field and in a magnetized medium using a purely general form-factor analysis. Although such an analysis is purely formal but still it can predict interesting effects, like the dependence of the neutrino self-energy on the angle between the neutrino propagation direction and the external magnetic field direction, in a magnetized medium. The fact that the neutrino self-energy is sensitive to the angle between the neutrino propagation direction and the magnetic field direction can have far reaching consequences. In the field of neutrino oscillations the angular dependence of the self-energy can modify the resonant level crossing condition of the neutrinos. The resonant condition itself becomes sensitive to the angle between the propagating neutrinos and the external magnetic field which can have an interesting astrophysical application like the asymmetric emission of neutrinos from a newly formed neutron star.

In the two following sections the fermion propagator in a magnetic field in vacuum and in a thermal medium will be discussed. In the remaining sections the topic of neutrino self-energy in a magnetic field and the effect of magnetic fields on neutrino oscillations will be discussed briefly. The chapter ends with a conclusion which summarizes the various things discussed in this chapter.

3.2 Propagators in vacuum

3.2.1 Furry propagator

There are two ways to write the propagator. The first is to start with the fermion field operator $\psi(X)$ written in terms of the spinor solutions and the creation and annihilation operators, and construct the time ordered product, as is usually done for finding the propagator of a free fermion field in the vacuum. The algebra is straight forward and yields the result

$$iS_B^V(X, X') = i \sum_N \int \frac{dp_0 dp_x dp_z}{(2\pi)^3} \frac{E_n + m}{p_0^2 - E_n^2 + i\epsilon} e^{-ip \cdot (X_y - X'_y)} \\ \times \sum_s U_s(y, N, \mathbf{p}_y) \overline{U}_s(y', N, \mathbf{p}_y), \quad (3.1)$$

where E_n is the positive root obtained from Eq. (2.16). The notation requires some explanation. Instead of x use has been made of X as the coordinates in the left hand side of the above equation. This style of writing is carried over from section 2.2. It is motivated by the fact that in calculations where we employ these propagators we often have to write down $\mathbf{p} \cdot \mathbf{X}_y \equiv p_x x + p_z z$ in the explicit form where the quantity x appears

as a coordinate and not as the 4-vector itself. The notation $S_B^V(X, X')$ specifies that the propagator is in vacuum and in presence of a uniform background magnetic field whose magnitude is \mathcal{B} . The spin sum appearing in Eq. (3.1) is given in Eq. (2.27). The resulting propagator is called the propagator in the Furry picture.

3.2.2 Schwinger propagator

Alternatively, one uses the propagators introduced by Schwinger [23]. Schwinger's calculation of the fermion propagator relies on a functional procedure and it is written in the form

$$iS_B^V(x, x') = \kappa(x, x') \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-x')} iS_B^V(p), \quad (3.2)$$

here x stands for the coordinate 4-vector as usual. $S_B^V(p)$ is expressed as an integral over a variable s , usually (though confusingly) called the 'proper time':

$$iS_B^V(p) = \int_0^\infty ds e^{\Phi(p, s)} G(p, s). \quad (3.3)$$

The quantities $\Phi(p, s)$ and $G(p, s)$ can be written in the following way :

$$\Phi(p, s) \equiv is \left(p_{\parallel}^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_{\perp}^2 - m^2 \right) - \epsilon|s|, \quad (3.4)$$

$$\begin{aligned} G(p, s) &\equiv \frac{e^{ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \left(\not{p}_{\parallel} + \frac{e^{-ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \not{p}_{\perp} + m \right) \\ &= (1 + i\sigma_z \tan(e\mathcal{B}s))(\not{p}_{\parallel} + m) + \sec^2(e\mathcal{B}s)\not{p}_{\perp}. \end{aligned} \quad (3.5)$$

In the above expressions $\sigma_z = i\gamma^1\gamma^2$. In a typical loop diagram, one therefore will have to perform not only integrations over the loop momenta, but also over the proper time variables. $iS_B(p)$ is manifestly translation invariant. Most of the calculations involving the Schwinger propagator use this translation invariant part only, while the other term $\kappa(x, x')$ appearing in the propagator in Eq. (3.2) remains irrelevant.

Not going into any detailed description of $\kappa(x, x')$ we can simply understand its necessity in the propagator. A propagator connects two points in space-time. In presence of a background gauge field the gauge transformation property of the fields of the charged fermions are different at two different space-time points. Unless there is some factor in the propagator which can connect these two fields at different space-time points with different gauge transformation properties, the calculations involving charged fermion propagators

will not be manifestly gauge invariant. In the limit the background magnetic field goes to zero Eq. (3.3) shows that $iS_B^V(p)$ reduces to the vacuum fermionic propagator. This indicates that $iS_B^V(p)$ in Eq. (3.2) do not carry any information about gauge invariance of the background gauge field, it is only the translation invariant modified version of the vacuum fermionic propagator. Therefore the gauge transformation property of the propagator must be related to $\kappa(x, x')$, and its presence is necessary for the overall gauge invariance of physical processes. The gauge transformed fields comes with phase factors where the phase depends upon the space-time point where the gauge transformation is made. The fermionic fields at two different space-time points will therefore have two different phase factors. To make a connection between them $\kappa(x, x')$ must also be some form of a phase. Conventionally it is named the phase-factor.

The phase factor is generally written as

$$\kappa(x, x') = \exp \{ieI(x, x')\} \quad (3.6)$$

where

$$I(x, x') = \int_{x'}^x d\xi^\mu \left[A_\mu(\xi) + \frac{1}{2} F_{\mu\nu}(\xi - x')^\nu \right] \quad (3.7)$$

where A_μ is the background gauge field, and $F_{\mu\nu}$ is the field strength tensor. From Eq. (3.7) we notice that the phase factor breaks the translation invariance of the propagator.

For a constant background field we can always write the gauge field as

$$A_\mu(\xi) = -\frac{1}{2} F_{\mu\nu} \xi^\nu + \partial_\mu \lambda(\xi), \quad (3.8)$$

where $\lambda(\xi)$ is an arbitrary well behaved function and depends upon our choice of gauge. Using the above relation in conjunction with Eq. (3.7) we can simplify the integration appearing in the phase factor as

$$I(x, x') = \int_{x'}^x d\xi^\mu \left[-\frac{1}{2} F_{\mu\nu} x'^\nu + \partial_\mu \lambda(\xi) \right]. \quad (3.9)$$

Using the constancy of the field strength tensor the above expression can be written as

$$I(x, x') = \frac{1}{2} x'^\mu F_{\mu\nu} x^\nu + \lambda(x) - \lambda(x'). \quad (3.10)$$

From Eq. (3.10) we can immediately see if we set $x = x'$, in other words if we integrate over any closed contour in space-time $I(x, x')$ vanishes. Thus $I(x, x')$ connecting two

points in space-time is independent of the path joining them, and as a result the phase factor of the Schwinger propagator joining the points x' and x in Eq. (3.6) is also path independent.

Utilizing the path independence of the phase factor of the propagator the general convention is to choose a straight line path connecting the two points x' and x . Points on this path are represented by

$$\xi^\mu = (1 - \zeta)x'^\mu + \zeta x^\mu, \quad (3.11)$$

where the parameter ζ ranges from 0 to 1. Using Eq. (3.7) and the straight line path given in Eq. (3.11) we get

$$\begin{aligned} I(x, x') &= \int_{x'}^x d\xi^\mu A_\mu(\xi) + \frac{\zeta}{2} \int_0^1 d\zeta (x^\mu - x'^\mu) F_{\mu\nu}(x^\nu - x'^\nu), \\ &= \int_{x'}^x d\xi^\mu A_\mu(\xi). \end{aligned} \quad (3.12)$$

Now using Eq. (3.8) for the gauge field we can retrieve Eq. (3.10).

From Eq. (3.10) it is clear that the phase factor is dependent on the form of the function $\lambda(\xi)$, that is to say the fermion propagator is dependent on the gauge in which the constant background magnetic field is specified. Suppose we are working in such a gauge that $\lambda(\xi) = 0$, and then we make a gauge transformation of the background field as

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda(\xi) \quad (3.13)$$

then from Eq. (3.2) it follows that the fermion propagator will transform as

$$iS_B^V(x, x') \rightarrow \exp(ie\lambda(x)) iS_B^V(x, x') \exp(-ie\lambda(x')). \quad (3.14)$$

under the gauge transformation. As because we are working in presence of a background gauge field, the fields of the charged particles and their propagators both become background gauge dependent. This is the reason why the phase factor arises in the expression of the propagator. A detailed study of the phase factor appears in appendix B.1.

3.3 Schwinger propagator in presence of a medium

In presence of a medium the propagators of the elementary particles get changed. There are various formulations of statistical field theory, like the imaginary time method,

the real time formalism, to name the most prominent ones. In the present work we will be using the real time formalism developed in the canonical approach [56]. Some of the important points of the above formalism is discussed in appendix C.

In this formalism it is not difficult to obtain the effects of a magnetized medium on the Schwinger propagator. In the real-time formalism, the 1-1 component of the propagator $iS(p)$ involving the time-ordered product³ can be written in terms of the free propagator $iS_0(p)$:

$$iS(p) = iS_0(p) - \eta_F(p \cdot u) [iS_0(p) - i\bar{S}_0(p)], \quad (3.15)$$

where

$$\bar{S}_0(p) = \gamma_0 S_0^\dagger(p) \gamma_0, \quad (3.16)$$

and $\eta_F(p \cdot u)$ contains the distribution function for particles and antiparticles:

$$\eta_F(p \cdot u) = \Theta(p \cdot u) f_F(p, \mu, \beta) + \Theta(-p \cdot u) f_F(-p, -\mu, \beta). \quad (3.17)$$

Here, Θ is the step function which takes the value +1 for positive values of its argument and vanishes for negative values of the argument, u^μ is the 4-vector denoting the center-of-mass velocity of the background plasma. Conventionally u is normalized in such a way that in the rest frame of the medium

$$u_\mu = (1, \mathbf{0}). \quad (3.18)$$

f_F denotes the Fermi-Dirac distribution function:

$$f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p \cdot u - \mu)} + 1}. \quad (3.19)$$

In a similar manner, the 1-1 component of the propagator in a magnetized plasma is given by [57, 58]

$$iS_B(p) = iS_B^V(p) - \eta_F(p \cdot u) [iS_B(p) - i\bar{S}_B(p)]. \quad (3.20)$$

In the Schwinger proper-time representation, this can also be written as an integral over the proper-time variable s :

$$iS_B(p) = \int_0^\infty ds e^{\Phi(p,s)} G(p, s) - \eta_F(p \cdot u) \int_{-\infty}^\infty ds e^{\Phi(p,s)} G(p, s), \quad (3.21)$$

³It should be mentioned here that other orderings also appear in the evaluation of general Green's functions. The other orderings give rise to other propagators as discussed in appendix C. We will not need those other propagators here.

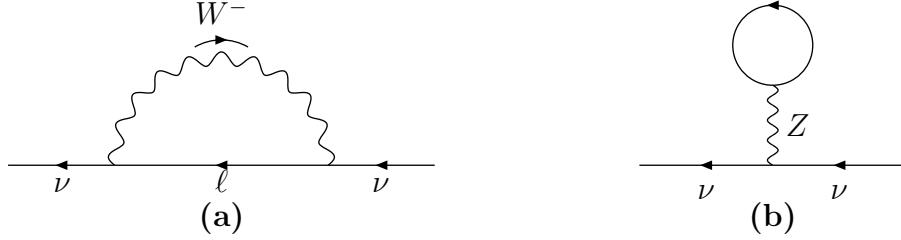


Figure 3.1: One-loop diagrams for neutrino self-energy in a magnetized medium. Diagram b is absent if the background contains only a magnetic field but no matter. For legends and related diagrams, see the caption of Fig. 1.1.

where $\Phi(p, s)$ and $G(p, s)$ are given by the expressions in Eq. (3.4) and Eq. (3.5). Symbolically the above equation can be written as

$$iS_B(p) = iS_B^V(p) + S_B^\eta(p) \quad (3.22)$$

where $iS_B^V(p)$ is the vacuum propagator in a uniform magnetic field and is given by Eq. (3.3). The other part $S_B^\eta(p)$ of the propagator carries the medium effect, and is given by

$$S_B^\eta(p) = -\eta_F(p \cdot u) \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} G(p, s). \quad (3.23)$$

It is straight forward to see that when $\mathcal{B} = 0$, the propagator in Eq. (3.3) reduces to

$$\begin{aligned} iS_0^V(p) &= \int_0^{\infty} ds \exp \left[is \left(p^2 - m^2 + i\epsilon \right) \right] (\not{p} + m) \\ &= i \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}, \end{aligned} \quad (3.24)$$

which is the normal vacuum Feynman propagator. In the same limit, the background dependent part reduces to

$$S_0^\eta(p) = -2\pi \delta(p^2 - m^2) \eta_F(p \cdot u) (\not{p} + m), \quad (3.25)$$

which is what we expect in a medium in absence of a magnetic field.

3.4 Neutrino self-energy

In chapter 1 the self-energy of the neutrino was discussed in a background magnetic field and an expression of the self-energy was supplied in Eq. (1.2). Here we take up

the issue and briefly discuss about the form of the self-energy expression [34]. Skipping detailed calculations using the Schwinger propagators a form-factor analysis of the neutrino self-energy is given in this section and its implications are highlighted in the case of neutrino oscillations.

It is already mentioned in chapter 1 that the simplest physical quantity where background magnetic field effects appear through virtual lines of charged particles is the self-energy of the neutrino. The 1-loop diagram for the self-energy is given in Fig. 3.1. In the present discussion we take the case of neutrinos in a medium with a magnetic field. From this general approach we will be able to find the form of the self-energy of the neutrino in absence of a medium also.

It is easy to see how the self-energy might be modified within a magnetized plasma. In the vacuum, the self-energy of a fermion has the general structure

$$\Sigma(q) = a\gamma^\mu q_\mu + b, \quad (3.26)$$

which is the most general form dictated by Lorentz covariance. Here, a and b are Lorentz invariant, and can therefore depend only on q^2 . In the presence of a homogeneous medium, the self-energy will involve the 4-vector u^μ . Further, if the medium contains a background magnetic field, the background field $F_{\mu\nu}$ also enters the general expression for the self-energy. These new objects, u^μ and $F_{\mu\nu}$, enter in two different ways. First, any form factor now can depend on more Lorentz invariants which are present in the problem. Second, the number of form factors also increases, since it is possible to write some more Lorentz covariant terms using u^μ and $F_{\mu\nu}$. There will in fact be a lot of form factors in the most general case. However, if we have chiral neutrinos as in the standard electroweak theory, the expression is not very complicated [34]:

$$\Sigma_B(q) = (a_1 q_\mu + b_1 u_\mu + a_2 q^\nu F_{\mu\nu} + b_2 u^\nu F_{\mu\nu} + a_3 q^\nu \tilde{F}_{\mu\nu} + b_3 u^\nu \tilde{F}_{\mu\nu}) \gamma^\mu L, \quad (3.27)$$

where L is the left-chiral projection operator, and

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}. \quad (3.28)$$

We first consider the self-energy when the background consists of a pure magnetic field, without any matter. Then all b -type form-factors disappear from the self-energy. The dispersion relation of neutrinos can then be obtained by the zeros of $\not{q} - \Sigma_B$, which gives

$$[(1 - a_1)q^\mu - a_2 q^\nu F_{\mu\nu} - a_3 q^\nu \tilde{F}_{\mu\nu}]^2 = 0. \quad (3.29)$$

It is interesting to note that the terms linear in the background field all vanish due to the antisymmetry of the field tensor. Moreover, the a_2a_3 term is also zero for a purely magnetic field. This gives

$$(1 - a_1)^2 q^\mu q_\mu + a_2^2 q^\nu q_\lambda F_{\mu\nu} F^{\mu\lambda} + a_3^2 q^\nu q_\lambda \tilde{F}_{\mu\nu} \tilde{F}^{\mu\lambda} + 2a_2 a_3 q^\nu q_\lambda F_{\mu\nu} \tilde{F}^{\mu\lambda} = 0. \quad (3.30)$$

The remaining terms can be most easily understood if we take the z -axis along the direction of the magnetic field. Then the only non-zero components of the tensor $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are given by

$$F_{12} = -F_{21} = \mathcal{B}, \quad \tilde{F}_{03} = -\tilde{F}_{30} = \mathcal{B}, \quad (3.31)$$

where we have adopted the convention

$$\epsilon_{0123} = +1. \quad (3.32)$$

Thus

$$\begin{aligned} q^\nu q_\lambda F_{\mu\nu} F^{\mu\lambda} &= -(q_x^2 + q_y^2) \mathcal{B}^2 = -q_\perp^2 \mathcal{B}^2, \\ q^\nu q_\lambda \tilde{F}_{\mu\nu} \tilde{F}^{\mu\lambda} &= (\Omega^2 - q_z^2) \mathcal{B}^2 = q_\parallel^2 \mathcal{B}^2, \end{aligned} \quad (3.33)$$

where the notations for parallel and perpendicular products were introduced in the chapter 1. The form factor a_1 can be set equal to zero by a choice of the renormalization prescription. So the dispersion relation is now a solution of the equation

$$q^2 - a_2^2 q_\perp^2 \mathcal{B}^2 + a_3^2 q_\parallel^2 \mathcal{B}^2 = 0, \quad (3.34)$$

which can also be written as

$$\Omega^2 = q_z^2 + \frac{1 + a_2^2 \mathcal{B}^2}{1 + a_3^2 \mathcal{B}^2} q_\perp^2. \quad (3.35)$$

Of course, this should not be taken as the solution for the neutrino energy, because the right hand side contains form factors which, in general, are functions of the energy and other things. But at least it shows that in the limit $\mathcal{B} \rightarrow 0$, the vacuum dispersion relation is recovered. If we retain the lowest order corrections in \mathcal{B} , we can treat the form-factors to be independent of \mathcal{B} and can write [34]

$$\Omega^2 = \mathbf{q}^2 + (a_2^2 - a_3^2) \mathcal{B}^2 q_\perp^2. \quad (3.36)$$

Calculation of this self-energy was performed by Erdas and Feldman [10] using the Schwinger propagator, where they also incorporated the modification of the W -propagator due to the magnetic field. Importantly, the W -propagator contains the same phase factor as discussed in appendix B.1. Therefore, the phase factors from the charged lepton and the W -lines are of the form $\kappa(x, x')\kappa(x', x)$. From appendix B.1 it is easy to see that this is equal to unity, and therefore the phase factors do not contribute to the final expression. Detailed calculations show that [10]

$$a_2^2 - a_3^2 = \left(\frac{eg}{2\pi M_W^2} \right)^2 \left(\frac{1}{3} \ln \frac{M_W}{m} + \frac{1}{8} \right), \quad (3.37)$$

where m is the mass of the charged lepton in the internal line. The above equation in conjunction with Eq. (3.36) gives us Eq. (1.2) appearing in the chapter 1.

Let us next concentrate on the terms which can occur only in a magnetized medium. In other words, we select out the terms which cannot occur if the neutrino propagates in a background of pure magnetic field without any material medium. This means that, apart from the term \not{q} which occurs also in the vacuum, we look for the terms which contain both u^μ and $F_{\mu\nu}$. Further, if the background field is purely magnetic in the rest frame of the medium, $u^\nu F_{\mu\nu} = 0$ since u has only the time component whereas the only non-zero components of $F_{\mu\nu}$ are spatial. Thus we are left with [59]:

$$\Sigma_B(q) = (a_1 q_\mu + b_1 u_\mu + b_3 u^\nu \tilde{F}_{\mu\nu}) \gamma^\mu L. \quad (3.38)$$

Once again, setting $a_1 = 0$ through a renormalization prescription, we can find the dispersion relation of the neutrinos in the form [59]:

$$\Omega = |\mathbf{q} - b_3 \mathbf{B}| + b_1 \approx |\mathbf{q}| - b_3 \hat{q} \cdot \mathbf{B} + b_1, \quad (3.39)$$

where \hat{q} is the unit vector along \mathbf{q} , and we have kept only the linear correction in the magnetic field. This form for the dispersion relation was first arrived at by D'Olivo, Nieves and Pal (DNP) [59] who essentially performed a calculation to the first order in the external field. As for the form factors, b_1 was known previously, obtained from the analysis of neutrino propagation in isotropic matter, i.e., without any magnetic field. The result was [35, 60, 61, 62]

$$b_1 = \sqrt{2G_F(n_e - n_{\bar{e}})} \times (y_e + \rho c_V), \quad (3.40)$$

where

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}, \quad (3.41)$$

n_e , $n_{\bar{e}}$ are the densities of electrons and positrons in the medium,

$$y_e = \begin{cases} 1 & \text{for } \nu_e, \\ 0 & \text{for } \nu \neq \nu_e, \end{cases} \quad (3.42)$$

and c_V is defined through the coupling of the electron to the Z -boson, whose Feynman rule is

$$-\frac{ig}{2 \cos \theta_W} \gamma_\mu (c_V - c_A \gamma_5). \quad (3.43)$$

In other words, in the standard model

$$c_V = -\frac{1}{2} + 2 \sin^2 \theta_W, \quad c_A = -\frac{1}{2}. \quad (3.44)$$

The contribution to b_3 from background electrons and positrons was calculated by DNP [59]. They obtained⁴

$$b_3 = -2\sqrt{2eG_F} \int \frac{d^3 p}{(2\pi)^3 2E} \frac{d}{dE} (f_e - f_{\bar{e}}) \times (y_e + \rho c_A), \quad (3.45)$$

where f_e and $f_{\bar{e}}$ are the Fermi distribution functions for electrons and positrons, and

$$E = \sqrt{\mathbf{p}^2 + m_e^2}. \quad (3.46)$$

Later authors have improved on this result in two different ways. Some authors [63] have included the contributions coming from nucleons in the background. Some others [57, 64] have used the Schwinger propagator and extended the results to all orders in the magnetic field.

3.5 The role of neutrino oscillations in a magnetized medium

Neutrino oscillation phenomenology in a medium has produced the important MSW effect [35, 36] which is now widely applied in the solar neutrino problem and also in the case of neutrinos coming from supernovas. But in most of the cases there is a background

⁴The authors of [59] used a convention in which $e < 0$. Here we present the result in the convention $e > 0$.

magnetic field and so the following discussion is an important part of neutrino oscillation phenomenology.

Calculation of neutrino self-energy has a direct consequence on neutrino mixing and oscillations. Of course neutrino oscillations require neutrino mixing and therefore neutrino mass. For the sake of simplicity, we discuss mixing between two neutrinos which we will call ν_e and ν_μ . The eigenstates will in general be called ν_1 and ν_2 , which are given by

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \quad (3.47)$$

We will denote the masses of the eigenstates by m_1 and m_2 , and assume that the neutrinos are ultra-relativistic. Then in the vacuum, the evolution equation for a beam of neutrinos will be given by

$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \frac{1}{2\omega} M^2 \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \quad (3.48)$$

where the matrix M^2 is given by

$$M^2 = \begin{pmatrix} -\frac{1}{2}\Delta m^2 \cos 2\theta & \frac{1}{2}\Delta m^2 \sin 2\theta \\ \frac{1}{2}\Delta m^2 \sin 2\theta & \frac{1}{2}\Delta m^2 \cos 2\theta \end{pmatrix}, \quad (3.49)$$

where $\Delta m^2 = m_2^2 - m_1^2$. In writing this matrix, we have ignored all terms which are multiples of the unit matrix, which affect the propagation only by a phase which is common for all the states.

In a non-trivial background, the dispersion relations of the neutrinos change, as discussed in Sec. 3.4. This adds new terms to the diagonal elements of the effective Hamiltonian in the flavor basis, which we denote by the symbol A . As a result, the matrix M^2 should now be replaced by

$$\widetilde{M}^2 = \begin{pmatrix} -\frac{1}{2}\Delta m^2 \cos 2\theta + A_{\nu_e} & \frac{1}{2}\Delta m^2 \sin 2\theta \\ \frac{1}{2}\Delta m^2 \sin 2\theta & \frac{1}{2}\Delta m^2 \cos 2\theta + A_{\nu_\mu} \end{pmatrix}, \quad (3.50)$$

where the extra contributions are in general different for ν_e and ν_μ . The eigenstates and eigenvalues change because of these new contributions. For example, the mixing angle now becomes $\tilde{\theta}$, given by

$$\tan 2\tilde{\theta} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta + A_{\nu_\mu} - A_{\nu_e}}. \quad (3.51)$$

In a pure magnetic field, the self-energies were shown in Eq. (3.36) and Eq. (3.37). The quantity m appearing in Eq. (3.37) is the mass of the charged lepton in the loop. Thus, for

ν_e , it is the electron mass whereas for ν_μ , it is the muon mass. Thus $A_{\nu_\mu} \neq A_{\nu_e}$. However, the difference appears in logarithmic form, and is presumably not very significant.

In a magnetized medium, however, the situation changes. The reason is that the medium contains electrons but not muons. Accordingly, the quantities A_{ν_e} and A_{ν_μ} can be very different, as seen by the presence of the term y_e in Eq. (3.45). If we take self-energy corrections only up to linear order in \mathcal{B} , as done in Eq. (3.39), we obtain

$$A_{\nu_\mu} - A_{\nu_e} = -\sqrt{2}G_F(n_e - n_{\bar{e}}) - 2\sqrt{2}eG_F\hat{\mathbf{q}} \cdot \mathbf{B} \int \frac{d^3p}{(2\pi)^3 2E} \frac{d}{dE}(f_e - f_{\bar{e}}). \quad (3.52)$$

The first term on the right side comes just from the background density of matter, and the second term is the magnetic field dependent correction. This quantity has been calculated for various combinations of temperature and chemical potential of the background electrons [57, 65, 66].

If the denominator of the right side of Eq. (3.51) becomes zero for some value of $A_{\nu_\mu} - A_{\nu_e}$, the value of $\tan 2\tilde{\theta}$ will become infinite. This is the resonant level crossing condition. This was first discussed in the context of neutrino oscillation in a matter background by Mikheev and Smirnov [36], where a particular value of density would ensure resonance. Presence of a magnetic field will modify this resonant density, as seen from Eq. (3.52). The modification will be direction dependent because of the factor $\hat{\mathbf{q}} \cdot \mathbf{B}$. Some early authors [65, 66] contemplated that, for large \mathcal{B} , the magnetic term might even drive the resonance. However, later it was shown [67] that the magnetic correction would always be smaller than the other term. So, if one considers values of \mathcal{B} which are so large that the last term in Eq. (3.52) is larger than the first term on the right hand side, it means that one must take higher order corrections in \mathcal{B} into account.

It should be noted that the type of corrections to the dispersion relation discussed in Sec. 3.4 appear from chiral neutrinos. Thus, they produce chirality-preserving modifications to neutrino oscillations. In addition, if the neutrino has a magnetic moment, there will be chirality-flipping modifications as well. Many of these modifications were analyzed in the context of the solar neutrino problem, and we do not discuss them here.⁵ As pointed out in chapter 1, they are not important for solar neutrinos, although may be important in other stellar objects like the neutron star where the magnetic fields are much larger.

⁵A recent paper on chirality-flipping oscillations is [68], where one can obtain references to earlier literature. Some early references are also found in [5] and [69].

An interesting possible consequence of neutrino oscillations have been discussed [70] in the context of high velocities of neutron stars. In chapter 2 the puzzle about the high velocities of the pulsars was addressed in terms of the asymmetric cross-section of the inverse beta decay process which compels asymmetric emission of neutrinos from the neutron stars. In the present situation the asymmetric emission of the neutrinos is generated by resonant conversion of the neutrinos inside the neutron star's body. In a material background containing electrons but not any other charged leptons, the cross-section of ν_e 's is greater than that of any other flavor of neutrino. If ν_e 's can oscillate resonantly to any other flavor, they can escape more easily from a star. In a proto-neutron star, the resonant density at an angle θ with the magnetic field occurs at a distance $R_0 + \delta \cos \theta$ from the center, where δ is a function of the magnetic field and specifies the deformation from a spherical surface. So this distance is direction dependent, as we have seen from the previous discussion on the resonant level crossing condition in a magnetized medium. Therefore the escape of neutrinos is also direction dependent, and the momentum carried away by them is not isotropic. The star would get a kick in the direction opposite to the net momentum of escaped neutrinos. This was suggested by Kusenko and Segrè [70], who estimated that the momentum imbalance is proportional to δ and can have a magnitude of around 1% for reasonable values of \mathcal{B} . Later authors [71] criticized their analysis and argued that the effect was overestimated by them, because the kick momentum vanishes in the lowest order in δ . A recent and detailed study [72] indicates that these criticisms may not be well-placed, and the kick momentum might indeed be proportional to the surface deformation parameter δ .

3.6 Conclusion

The main points discussed in this chapter are related to the propagator of a charged particle and the neutrino self-energy in a magnetic field and in a magnetized medium. The Schwinger propagator acts like the central point related to which all the discussion on this chapter follows. This chapter also acts as a precursor for the rest of the material in this thesis because what follows heavily relies on the expression of the Schwinger propagator in a magnetized medium.

While discussing about the Schwinger propagator due attention has been paid to the phase factor accompanying it. It has been pointed out that the origin and the purpose of

the phase factor is related to the gauge degree of freedom of the background gauge field. Subsequently the methods of statistical field theory has been utilized to write down the expression of the Schwinger propagator in a thermal medium.

As a magnetic field interacts with the intermediate virtual charged particles, so the neutrino self-energy also gets modified from its vacuum value in presence of a magnetic field. In the section on neutrino self-energy the expression of the self-energy is derived from a purely general idea based on a form-factor analysis. From the discussion on the neutrino self-energy it is evident that the general form-factor analysis in a magnetized medium yields considerable information. The power of such an analysis is felt when it predicts the dependence of the self-energy on the angle between neutrino propagation direction and the external magnetic field in a magnetized medium without the help of any actual loop calculation.

Section 3.5 discusses about neutrino oscillations in a magnetized medium. The important question about the effect of an external magnetic field on the resonant conversion of neutrinos in a medium has been addressed, which is a very important topic in astrophysics. In this section another attempt has been made to explain the puzzle about the high velocities of the proto-neutron stars by the mechanism of resonant conversion of neutrinos from one flavour to the other. In literature this type of mechanism by which the proto-neutron stars get a kick in one specific direction due to asymmetric emission of neutrinos is often called the ‘neutrino rocket mechanism’.

CHAPTER 4

Electromagnetic interactions in a magnetized medium

4.1 Introduction

This chapter focuses on actual calculations done using the Schwinger propagator. The following section will focus on the issue of the effective electromagnetic vertex of the neutrinos in presence of a magnetized medium. But before going into the techniques of calculation a brief introduction to the topic of electromagnetic vertex of neutrinos is presented here.

Previously a lot of work has been done concerning the neutrino-neutrino-photon vertex in the presence of a background magnetic field [34]. The vertex arises from the diagrams of Fig. 1.1, which contain internal W -lines. In addition, there is a diagram mediated by the Z -boson, as shown in Fig. 4.1. For phenomenological purposes, we require the electromagnetic vertex of neutrinos only in the leading order in Fermi constant. It should be realized that in this order, the diagram of Fig. 1.1b does not contribute at all, since it has two W -propagators. The remaining diagrams, shown in Fig. 1.1a and Fig. 4.1, can both be represented in the form shown in Fig. 4.2, where an effective 4-fermi vertex has been used. The effective 4-fermi interaction can be written as,

$$\mathcal{L}_{\text{eff}} = -\sqrt{2}G_F \left[\bar{\psi}_{(\nu)} \gamma^\lambda L \psi_{(\nu)} \right] \left[\bar{\psi}_{(\ell)} \gamma_\lambda (g_V + g_A \gamma_5) \psi_{(\ell)} \right], \quad (4.1)$$

where $L = \frac{1}{2}(1-\gamma_5)$ is the left handed projection operator. If the neutrino and the charged lepton belong to different generations of fermions, this effective Lagrangian contains only the neutral current interactions, and in that case g_V and g_A are equivalent to c_V and $-c_A$ defined in Eq. (3.44). On the other hand, if both ν and ℓ belong to the same generation, we should add the charged current contribution as well, and use

$$g_V = c_V + 1, \quad g_A = -(c_A + 1). \quad (4.2)$$

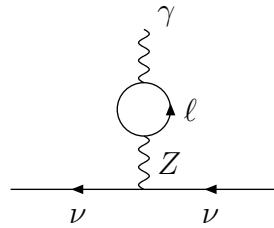


Figure 4.1: Z -photon mixing diagram contributing to the neutrino electromagnetic vertex.

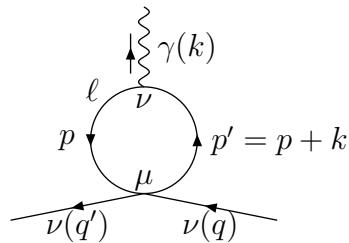


Figure 4.2: The neutrino electromagnetic vertex in the leading order in the Fermi constant.

Many processes involving neutrinos and photons have been calculated using the 4-fermi Lagrangian of Eq. (4.1). The calculations simplify in this limit for various reasons. First, we do not have to use the momentum dependence of the gauge boson propagators. Second, since two charged lepton lines form a loop in Fig. 4.2, the phase factors of the propagators, as discussed in appendix B.1, cancel each other.

The background magnetic field, can give rise to many physical processes which are forbidden in the vacuum. One such process is the decay of a photon into a neutrino-antineutrino pair:

$$\gamma \rightarrow \nu + \bar{\nu}. \quad (4.3)$$

This was calculated using the Schwinger propagator in some very early papers [73, 26]. Assuming two generations of fermions, the decay rate was found to be

$$\Gamma = \frac{\alpha^2 G_F^2}{48\pi^3 \omega} |\varepsilon^\mu k^\nu \tilde{F}_{\mu\nu}|^2 |\mathcal{M}_e - \mathcal{M}_\mu|^2, \quad (4.4)$$

where ε^μ , k^μ and ω are the polarization vector, the momentum 4-vector and the energy of the initial photon, and the quantity \mathcal{M}_ℓ was evaluated in various limits by these authors. $\tilde{F}_{\mu\nu}$ is the dual of the electromagnetic field strength tensor. For example, if $\omega \ll m_\ell$, they found

$$\mathcal{M}_\ell = \frac{\omega^2}{e\mathcal{B}} \sin^2 \theta \times \begin{cases} \frac{2}{15} \left(\frac{e\mathcal{B}}{m_\ell^2} \right)^3 & \text{for } e\mathcal{B} \ll m_\ell^2, \\ \frac{1}{3} \left(\frac{e\mathcal{B}}{m_\ell^2} \right) & \text{for } e\mathcal{B} \gg m_\ell^2, \end{cases} \quad (4.5)$$

where θ is the angle between the photon momentum and the magnetic field. No matter which neutrino pair the photon decays to, both charged leptons appear in the decay rate because of the loop in Fig. 4.1. In the calculation only the axial couplings of the charged leptons contribute to the amplitude.

A related process is Cherenkov radiation from neutrinos:

$$\nu \rightarrow \nu + \gamma. \quad (4.6)$$

Again, this is a process forbidden in the vacuum. But a background magnetic field modifies the photon dispersion relation, and so this process becomes feasible. The rate of this process has been calculated by many authors [73, 74, 28, 75], and all of them do not get the same result. According to Ref. [75], the rate for the process is given by

$$\Gamma = \frac{\alpha G_F^2}{8\pi^2} (g_V^2 + g_A^2) (e\mathcal{B})^2 \Omega \sin^2 \theta F(\Omega^2 \sin^2 \theta / e\mathcal{B}), \quad (4.7)$$

where Ω is the initial neutrino energy and θ is the angle between the initial neutrino momentum and the background field. For large magnetic fields satisfying the condition $e\mathcal{B} \gg \omega^2 \sin^2 \theta$, the function F is given by

$$F(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{5x^3}{24} + \frac{7x^4}{60} + \dots. \quad (4.8)$$

The modification of this process in the presence of background matter has also been calculated [76].

Another process that has been discussed is the radiative neutrino decay

$$\nu_a \rightarrow \nu_b + \gamma. \quad (4.9)$$

Unlike the previous processes, this can occur in the vacuum as well when the neutrinos have mass and mixing. However, a background magnetic field adds new contributions to

the amplitude, and the rate can be enhanced. Gvozdev, Mikheev and Vasilevskaya [77] calculated the rate of this decay in a variety of situations depending on the field strength and the energy of the initial neutrino. For a strong magnetic field ($\mathcal{B} \gg \mathcal{B}_e$), they found the decay rate of an ultra-relativistic neutrino of energy Ω to be

$$\Gamma = \frac{2\alpha G_F^2}{\pi^4} \frac{m^6}{\Omega} \left(\frac{\mathcal{B}}{\mathcal{B}_e} \right)^2 |U_{ae} U_{be}^*|^2 J(\Omega \sin \theta / 2m_e), \quad (4.10)$$

where U is the leptonic mixing matrix, θ is the angle between the magnetic field and the neutrino momentum, m is the electronic mass, and

$$J(z) = \int_0^z dy (z-y) \left(\frac{1}{y\sqrt{1-y^2}} \tan^{-1} \frac{y}{\sqrt{1-y^2}} - 1 \right)^2. \quad (4.11)$$

The curious feature of this result is that this is independent of the initial and the final neutrino masses.

The form for the 4-fermi interaction in Eq. (4.1) suggests that the neutrino electromagnetic vertex function Γ_μ can be written as [78]:

$$\Gamma_\mu = -\frac{\sqrt{2}G_F}{e} \gamma^\nu L (g_V \Pi_{\mu\nu} + g_A \Pi_{\mu\nu}^5). \quad (4.12)$$

Here, the term $\Pi_{\mu\nu}$ is exactly the expression for the vacuum polarization of the photon, and appears from the vector interaction in the effective Lagrangian. The other term, $\Pi_{\mu\nu}^5$, differs from $\Pi_{\mu\nu}$ in that it contains an axial coupling from the effective Lagrangian. This equality is valid even when one has a magnetic field and a material medium as the background. Thus, the calculation of the photon self-energy in a background magnetic field in matter can give us information about the neutrino electromagnetic vertex in the same situation.

Here we concentrate on calculations done in presence of a medium with a uniform background magnetic field. We consider various cases where the charged particles affected by the magnetic fields remain as virtual particles in the Feynman diagrams of the respective processes. The techniques of the calculation involve the use of the propagators introduced in chapter 3. This chapter discusses the electromagnetic vertex function of neutrinos and consequently the structure of the two second rank tensors $\Pi_{\mu\nu}$ and $\Pi_{\mu\nu}^5$ which are necessary to calculate it, as is evident from Eq. (4.12). $\Pi_{\mu\nu}$ has been calculated in a magnetic field and in a magnetized medium by various authors [25, 29, 30]. In the following section only the relevant portions of the results are presented briefly. The main

discussion is around the calculation of $\Pi_{\mu\nu}^5$ in presence of a magnetized medium. For the sake of comparison the results of the calculations has been given for various backgrounds as vacuum, medium, magnetic field and obviously a magnetized medium. Some comments on neutrino-photon scattering in a background magnetic field, which is highly suppressed in vacuum, is provided in subsection 4.2.1. Section 4.3 discusses the concept of the effective electric charge of a neutrino from a quantum field theoretical point of view. As neutrinos do not couple to photons in the tree level in the standard model so naturally if the neutrinos acquire some electric charge it must be an effective one. This effective charge of neutrinos in a magnetized medium has been calculated to odd orders in the background magnetic field in section 4.3. Section 4.4 describes the methods for handling ultraviolet divergences in calculations involving background magnetic fields. This chapter concludes with a general discussion on the various ideas exposed in the following sections.

4.2 Neutrino-photon scattering and the electromagnetic vertex of neutrinos

In this section we will discuss effective neutrino photon interactions in a magnetized medium [32, 31], and the constituents of the neutrino photon electromagnetic vertex. But before taking up the issue of the $\nu\nu\gamma$ vertex some comments on the neutrino-photon scattering in a background magnetic field follows.

4.2.1 Some comments on neutrino-photon scattering

The cross-section of neutrino-photon scattering is highly suppressed in the standard model due to Yang's theorem [79], which makes the scattering cross-section vanish to the first order of the Fermi coupling G_F . But in presence of a magnetic field, neutrino-photon scattering can occur and to orders of G_F the cross-section has been calculated [80]. In the following paragraphs of this subsection a brief overview on neutrino photon scattering in a magnetic field background is supplied.

Gell-Mann showed [81] that the amplitude of the reaction

$$\gamma + \nu \rightarrow \gamma + \nu \quad (4.13)$$

is exactly zero to order G_F because by Yang's theorem [79, 82] two photons cannot couple to a $J = 1$ state. In the standard model, therefore, amplitude of the above process appears only at the level of $1/M_W^4$ and as a result the cross-section is exceedingly small [83].

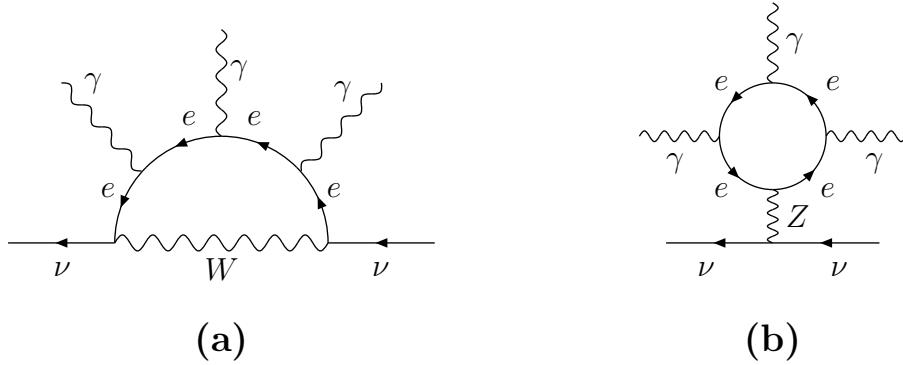


Figure 4.3: The 1-loop effective vertex for two neutrinos and three photons.

But there is no such restriction on the coupling of three photons with neutrinos as,

$$\gamma + \nu \rightarrow \gamma + \gamma + \nu. \quad (4.14)$$

The cross-section of the above process can be calculated from the effective Lagrangian proposed by Dicus and Repko [84]. The diagrams for the two neutrino three photon interaction are shown in Fig. 4.3 where Fig. 4.3a shows the contribution from the W exchange diagram and Fig. 4.3b shows the contribution from Z exchange. Denoting the photon field tensor as $\bar{F}_{\mu\nu}$ and the neutrino fields by $\psi_{(\nu)}$, and integrating out the particles in the loop the effective Lagrangian comes out as

$$\mathcal{L}_{\text{eff}} = \frac{G_F e^3 (c_V + 1)}{\sqrt{2} 360 \pi^2 m^4} \left[5(N_{\mu\nu} \bar{F}^{\mu\nu})(\bar{F}_{\lambda\rho} \bar{F}^{\lambda\rho}) - 14 N_{\mu\nu} \bar{F}^{\nu\lambda} \bar{F}_{\lambda\rho} \bar{F}^{\rho\mu} \right], \quad (4.15)$$

where c_V was defined in Eq. (3.44), m is the electronic mass, and

$$N_{\mu\nu} = \partial_\mu (\bar{\psi}_{(\nu)} \gamma_\nu L \psi_{(\nu)}) - \partial_\nu (\bar{\psi}_{(\nu)} \gamma_\mu L \psi_{(\nu)}). \quad (4.16)$$

For energies much smaller than the electron mass, this can be used as an effective Lagrangian to calculate various processes involving photons and neutrinos in the presence of a background magnetic field $F_{\mu\nu}$. For this, we simply have to write

$$\bar{F}_{\mu\nu} = f_{\mu\nu} + F_{\mu\nu}, \quad (4.17)$$

where now $f_{\mu\nu}$ is the dynamical photon field, and look for the terms involving $F_{\mu\nu}$. For example, Shaisultanov [80] calculated the rate of $\gamma\gamma \rightarrow \nu\bar{\nu}$ in a background field. Eq. (4.15) shows that in the lowest order, the amplitude for involving ν_e 's would be proportional to

$$\frac{G_F B}{m^4} \sim \frac{\mathcal{B}}{M_W^2 m^2 \mathcal{B}_e}, \quad (4.18)$$

where \mathcal{B}_e is the value of the magnetic field defined in Eq. (1.7). Since the amplitude without any magnetic field [83] is of order $1/M_W^4$, it follows that the background field increases the amplitude by a factor of order $(M_W/m)^2 \mathcal{B}/\mathcal{B}_e$, or the rate by a factor $(M_W/m_e)^4 (\mathcal{B}/\mathcal{B}_e)^2$. Later calculations [85] have extended these results by including other processes obtained by crossing, like $\nu\bar{\nu} \rightarrow \gamma\gamma$ and $\nu\gamma \rightarrow \nu\gamma$. To obtain higher \mathcal{B} terms in these cross-sections, one needs the effective Lagrangian containing higher order terms in the electromagnetic field strength. Such an effective Lagrangian has been derived by Gies and Shaisultanov [86].

Alternatively, the amplitudes can be calculated using the Schwinger propagator for charged leptons. Such calculations for $\gamma\gamma \rightarrow \nu\bar{\nu}$ were done some time ago [87, 55]. One of the important features of this calculation is that in the 4-fermi limit, the diagram contains three electron propagators. In such situations, the phase factor $\Psi(x, x')$ appearing in the Schwinger propagator of Eq. (3.2) cannot be disregarded⁶. In the calculation, only the linear term in \mathcal{B} was retained in the amplitude so that the results are valid only for small magnetic fields. However, since no effective Lagrangian was used, the results are valid even when the energies of the neutrinos and/or the photons are comparable to, or greater than, the electron mass. Later authors [85] reported some mistakes in this calculation and corrected them.

The previous discussions about neutrino-photon scattering relied on effective Lagrangians and the use of Schwinger propagator to calculate a 3-point function, but aside these scattering phenomena there are a lot of phenomena, discussed in the introduction of this chapter, which involves the $\nu\nu\gamma$ vertex in a background magnetic field. To calculate the rates of these processes one has to calculate the effective electromagnetic vertex of neutrinos in a background magnetic field and in a magnetized medium. This section is primarily concerned about the electromagnetic vertex of the neutrinos in a magnetized medium.

For simplicity we consider the background temperature and neutrino momenta to be small compared to the masses of the W and Z bosons. We can, therefore, neglect the momentum dependence in the W and Z propagators, which is justified if we are performing a calculation to the leading order in the Fermi constant, G_F . In this limit the 4-fermi interaction is given by Eq. (4.1) written in the beginning of this chapter. Moreover if we

⁶A detailed discussion on the phase factor of the Schwinger propagator is presented in subsection 3.2.2 of chapter 3 and appendix B.1.

restrict the temperature of the system such that muons and taons are not produced in the medium then in the leptonic part of the Lagrangian in Eq. (4.1), $\psi_{(\ell)}$ stands for electrons alone. For electron neutrinos,

$$g_V = \frac{1}{2} + 2 \sin^2 \theta_W,$$

$$g_A = -\frac{1}{2}.$$

For muon and tau neutrinos,

$$g_V = -\frac{1}{2} + 2 \sin^2 \theta_W,$$

$$g_A = \frac{1}{2}.$$

With the interaction Lagrangian as given in Eq. (4.1) the $\nu\nu\gamma$ vertex, as shown in Fig. 4.2, can be written in terms of two tensors. The vector-vector amplitude $\Pi_{\mu\nu}(k)$ and the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$. This fact is symbolically written down in Eq. (4.12). The vector-vector amplitude term $\Pi_{\mu\nu}(k)$ turns out to be:

$$i\Pi_{\mu\nu}(k) = (-ie)^2(-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr} [\gamma_\mu iS(p)\gamma_\nu iS(p')] . \quad (4.19)$$

Here and henceforth $p' = p + k$. From the above equation it can be seen that $\Pi_{\mu\nu}(k)$ is the photon vacuum polarization tensor. The axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ comes out as:

$$i\Pi_{\mu\nu}^5(k) = (-ie)^2(-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr} [\gamma_\mu \gamma_5 iS(p)\gamma_\nu iS(p')] . \quad (4.20)$$

Both tensors are obtained by calculating the Feynman diagram given in Fig. 4.2. The following subsections discusses about the second rank tensors $\Pi_{\mu\nu}(k)$ and $\Pi_{\mu\nu}^5(k)$.

4.2.2 The photon vacuum polarization tensor $\Pi_{\mu\nu}(k)$ in different backgrounds

The calculation of the photon vacuum polarization tensor in a magnetized medium has been done by previous authors [29, 30]. In this subsection only a brief summary of the previously known results is reproduced. The form of $\Pi_{\mu\nu}(k)$ to one-loop is given in Eq. (4.19) and it is also pointed out that $\Pi_{\mu\nu}(k)$ is a necessary quantity for understanding the neutrino electromagnetic vertex.

The form of $\Pi_{\mu\nu}(k)$ is restricted by two symmetries. Owing to gauge invariance, $\Pi_{\mu\nu}(k)$ satisfies the conditions

$$k^\mu \Pi_{\mu\nu}(k) = 0, \quad k^\nu \Pi_{\mu\nu}(k) = 0 . \quad (4.21)$$

In addition, Bose symmetry implies

$$\Pi_{\mu\nu}(k) = \Pi_{\nu\mu}(-k). \quad (4.22)$$

These conditions restrict the form of the possible tensor structure of the vacuum polarization tensor. The general form of $\Pi_{\mu\nu}(k)$ in various backgrounds is dictated by Eq. (4.21) and Eq. (4.22) as discussed below.

In vacuum

In the vacuum, the tensor $\Pi_{\mu\nu}(k)$ depends only on the momentum vector k . Thus, the most general form for $\Pi_{\mu\nu}(k)$ is given as,

$$\Pi_{\mu\nu}(k) = \Pi(k^2) [k^2 g_{\mu\nu} - k_\mu k_\nu]. \quad (4.23)$$

This form satisfies both the conditions given in Eq. (4.21) and Eq. (4.22).

In a medium

In a medium the expression of $\Pi_{\mu\nu}(k)$ differs from that in the vacuum. Although $\Pi_{\mu\nu}(k)$ still has to satisfy Eq. (4.21) and Eq. (4.22), the form given in Eq. (4.23) does not follow. This is because $\Pi_{\mu\nu}(k)$ can now depend, apart from the momentum vector k^μ , on various vectors or tensors which characterize the background medium. Even for a homogeneous and isotropic medium, there is an extra vector in the form of the velocity of its center of mass, u^μ . In general in place of k^2 the Lorentz scalars will now depend upon

$$\omega = k \cdot u, \quad (4.24)$$

$$K = \sqrt{\omega^2 - k^2}. \quad (4.25)$$

The above expressions of ω and K reduce to the energy and momentum of the photon in the rest frame of the thermal medium where u^μ is given as in Eq. (3.18).

In presence of a medium without any magnetic field the most general expression of the polarization tensor can be written in terms of form factors along with the new tensors constructed out of u^μ and the ones we already had in absence of a medium as [88],

$$\Pi_{\mu\nu}(K, \omega) = \Pi_T T_{\mu\nu} + \Pi_L L_{\mu\nu} + \Pi_P P_{\mu\nu}. \quad (4.26)$$

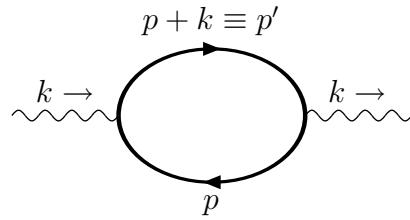


Figure 4.4: One-loop diagram for vacuum polarization.

Here

$$T_{\mu\nu} = \tilde{g}_{\mu\nu} - L_{\mu\nu}, \quad (4.27)$$

$$L_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2}, \quad (4.28)$$

$$P_{\mu\nu} = \frac{i}{K} \varepsilon_{\mu\nu\alpha\beta} k^\alpha u^\beta, \quad (4.29)$$

with

$$\tilde{u}_\mu = \tilde{g}_{\mu\rho} u^\rho, \quad (4.30)$$

and

$$\tilde{g}_{\mu\rho} = g_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}. \quad (4.31)$$

In Eq. (4.26) all form-factors are functions of ω , K and other parameters as chemical potential μ and inverse temperature β . In Eq. (4.26), Π_P is nonzero only if the background medium or the interaction Lagrangian of the photon break both **P** and **CP** symmetries [88].

In presence of a background magnetic field

The vacuum polarization of the photon in a magnetic field has also been calculated [25] up to one loop but to all orders in the external field and the expression of $\Pi_{\mu\nu}(k)$ in our convention, as discussed in section 1.3, is given by

$$\Pi_{\mu\nu}(k) = \frac{e^3 \mathcal{B}}{(4\pi)^2} \left[(g_{\mu\nu} k^2 - k_\mu k_\nu) N_0 - (g_{\mu\nu}^{\parallel} k_{\parallel}^2 - k_\mu^{\parallel} k_\nu^{\parallel}) N_{\parallel} - (g_{\mu\nu}^{\perp} k_{\perp}^2 + k_\mu^{\perp} k_\nu^{\perp}) N_{\perp} \right]. \quad (4.32)$$

Here N_0 , N_{\parallel} and N_{\perp} are the form-factors, which are functions of \mathcal{B} , k_{\parallel}^2 , and k_{\perp}^2 .

In presence of a magnetized medium

The discussion on $\Pi_{\mu\nu}(k)$ in presence of a magnetized medium is presented with some more details than the previous ones. This is because the form of $\Pi_{\mu\nu}(k)$ is important for the expression of the effective charge of the neutrinos in a magnetized medium. It also sets the stage for the calculation of $\Pi_{\mu\nu}^5(k)$ in a magnetized medium.

At the 1-loop level, the vacuum polarization tensor arises from the diagram in Fig. 4.4. To evaluate this diagram, one needs to use the electron propagator within a thermal medium in the presence of a background electromagnetic field. Rather than working with the complicated expression for a general background field, we will specialize to the case of a uniform magnetic field. Once this is assumed, the field can be taken in the z -direction without any further loss of generality.

The amplitude of the 1-loop diagram of Fig. 4.4 is given by the expression in Eq. (4.19). Only in this case the propagators appearing in Eq. (4.19) like $S(p)$ and $S(p')$ must be replaced by the appropriate ones suitable in a magnetized medium, $S_B(p)$ and $S_B(p')$ as introduced in chapter 3, Eq. (3.22). As the loop is a 2-point function the phase factor drops out of the calculation, as is discussed in appendix B.1. From Eq. (3.22), we see that there are two terms in the propagators $S_B(p)$ in a magnetized medium, the vacuum part and the other part which involves background matter distribution. In the trace appearing in the right hand side of Eq. (4.19) there appears a product of two propagators $S_B(p)$ and $S_B(p')$ with other quantities in between. This product will give rise to four quantities of which one term will contain two factors of S_B^V s from the two propagators. It has no importance as far as the background effects are concerned. There will be one term containing two factors of S_B^η s, which will contribute only to the absorptive part of the vacuum polarization. There will be two terms in the trace each containing a factor of S_B^V and a factor of S_B^η . These terms contribute to the present calculation, as they are important for calculating the effective charge of the neutrinos in a magnetized medium. The relevant part of $\Pi_{\mu\nu}(k)$ is therefore made up of the two terms inside the trace each containing one vacuum part and one thermal part of the total electron propagator. With this introduction the expression of $\Pi_{\mu\nu}(k)$ is written as

$$\Pi_{\mu\nu}(k) = -ie^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma_\mu S_B^\eta(p) \gamma_\nu iS_B^V(p') + \gamma_\mu iS_B^V(p) \gamma_\nu S_B^\eta(p') \right]. \quad (4.33)$$

Using the form of the propagators from Eq. (3.3) and Eq. (3.23) and manipulating under the integral signs [30], the 1-1-component⁷ of the polarization tensor to odd powers in the external magnetic field can be written as [30],

$$\begin{aligned}\Pi_{\mu\nu}(k) &= 4ie^2\varepsilon_{\mu\nu\alpha||\beta}k^\beta \int \frac{d^4p}{(2\pi)^4} \eta_-(p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} \\ &\times \left[p^{\tilde{\alpha}||} \tan(e\mathcal{B}s) + p'^{\tilde{\alpha}||} \tan(e\mathcal{B}s') - \frac{\tan(e\mathcal{B}s) \tan(e\mathcal{B}s')}{\tan(e\mathcal{B}(s+s'))} (p+p')^{\tilde{\alpha}||} \right].\end{aligned}\quad (4.34)$$

The above form of the vacuum polarization tensor is valid in the rest frame of the background medium. $\eta_-(p_0)$ contains the information about the distribution of electrons and positrons in the background and is given by

$$\eta_-(p_0) = \eta_F(p_0) - \eta_F(-p_0), \quad (4.35)$$

where $\eta_F(p_0)$ is specified in Eq. (3.17). In the above equation the argument of the η_F function is not $p \cdot u$ but only the energy component p_0 , because in the rest frame of the medium $p \cdot u = p_0$ due to the form of u in the rest frame, given in Eq. (3.18). From the form of $\Pi_{\mu\nu}(k)$ as given above it is seen that it is antisymmetric in its tensor indices.

4.2.3 The axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ in different backgrounds

The form of the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ to one-loop is supplied in Eq. (4.20). In this regard it must be noted that the importance of this second rank tensor arises from the fact that it is an important constituent of the neutrino electromagnetic vertex as given in Eq. (4.12). The neutrino electromagnetic vertex to one-loop is depicted in Fig. 4.2. While discussing about $\Pi_{\mu\nu}^5(k)$ it should be remembered that for the electromagnetic vertex i.e., the vertex in the Feynman diagram where the external photon couples to the fermion loop in Fig. 4.2, we have the gauge invariance condition,

$$k^\nu \Pi_{\mu\nu}^5(k) = 0. \quad (4.36)$$

In the following paragraphs the form of $\Pi_{\mu\nu}^5(k)$ in various backgrounds is discussed.

⁷The 1-1-component of the propagator is discussed in appendix C, where quantum statistical field theory is discussed.

In vacuum

$\Pi_{\mu\nu}^5(k)$ violates parity due to the presence of γ_5 in its expression as given in Eq. (4.20).

In accordance with this fact the available vectors and tensors at hand are the following,

$$k_\mu, g_{\mu\nu} \text{ and } \epsilon_{\mu\nu\lambda\sigma}. \quad (4.37)$$

The two point axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ can be expanded in a basis constructed out of the above tensors. Given the axial-tensor like parity structure of $\Pi_{\mu\nu}^5(k)$ the relevant tensors at hand are $\epsilon_{\mu\nu\lambda\sigma} k^\lambda k^\sigma$ and $\epsilon_{\mu\nu\lambda\sigma} g^{\lambda\sigma}$ both of which are zero. So in vacuum $\Pi_{\mu\nu}^5(k)$ vanishes.

In a medium

On the other hand, in a medium in absence of any magnetic field, we have an additional vector u^μ , i.e the velocity of the centre of mass of the medium. Therefore the axialvector-vector amplitude can be expanded in terms of form-factors along with the new tensors constructed out of u^μ and the ones we already had in absence of a medium. A second rank tensor constructed out of them can be $\varepsilon_{\mu\nu\alpha\beta} u^\alpha k^\beta$, [24, 5] which would verify the current conservation condition for the two point function.

In presence of a background magnetic field

In this case neither **C** or **P** but **CP** is a symmetry, first we look at the **CP** transformation properties of the axialvector-vector amplitude⁸. As the $Z_\mu - A_\nu$ mixing Lagrangian is of the form,

$$\mathcal{L}_{Z-A} \propto Z^\mu(k) \Pi_{\mu\nu}^5(k) A^\nu(-k), \quad (4.38)$$

we can specify the **CP** transformation property of $\Pi_{\mu\nu}^5(k)$ knowing the **CP** transformation properties of the fields Z_μ and A_ν . Under a **CP** transformation the various components of the tensor transforms as,

$$\begin{aligned} \text{CP : } & \Pi_{00}^5 \rightarrow \Pi_{00}^5, \\ \text{CP : } & \Pi_{ij}^5 \rightarrow \Pi_{ij}^5, \\ \text{CP : } & \Pi_{0i}^5 \rightarrow -\Pi_{0i}^5. \end{aligned} \quad (4.39)$$

⁸This is a preliminary attempt to get some idea about the tensorial basis of $\Pi_{\mu\nu}^5(k)$. Strictly speaking this attempt to use **CP** transformation properties of the axialvector-vector amplitude must be incomplete because we know that the **CP** symmetry is also broken in nature.

At first, out of all the tensors available in the present situation, a set of mutually orthogonal 4-vectors are constructed with specific **CP** transformation properties. In the next step these different 4-vectors are combined in pairs to produce a second rank tensor which has similar **CP** transformation properties to that of $\Pi_{\mu\nu}^5(k)$. As the 4-vectors whose combinations give the second rank tensors were mutually orthogonal to start with, the second rank tensors are also mutually orthogonal. The basic task in this section is to list all the second rank tensors which have similar **CP** transformation properties to that of $\Pi_{\mu\nu}^5(k)$.

Before starting the classification of the various second rank tensors at hand which have similar **CP** transformation properties as that of $\Pi_{\mu\nu}^5(k)$, some preliminary remarks about the form-factor analysis is in order. In the beginning of this section we have seen that $\Pi_{\mu\nu}^5(k)$ is zero in the vacuum. As a consequence it can be inferred that $\Pi_{\mu\nu}^5(k)$ calculated in the presence of a background uniform magnetic field vanishes as the magnetic field goes to zero. In a uniform background magnetic field, the vectors and tensors at hand are

$$F_{\mu\nu}, \quad \tilde{F}_{\mu\nu}, \quad k_{\parallel}^{\mu}, \quad k_{\perp}^{\mu}. \quad (4.40)$$

$F_{\mu\nu}$ is the field strength tensor for the uniform background magnetic field and $\tilde{F}^{\mu\nu}$ is its dual given as $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. For a magnetic field directed along the z -axis the expressions of F_{12} , F_{21} , F_{03} , F_{30} are given in Eq. (3.31) in chapter 3. All other components of $F_{\mu\nu}$ zero. The sign of the 4-dimensional totally antisymmetric tensor is specified in Eq. (3.32). Now due to Lorentz invariance the form-factors multiplying the basis tensors can be functions of k^2 , k_{\parallel}^2 , k_{\perp}^2 and even functions of the background magnetic field as $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}$ and other higher powers. Terms containing an odd number of $F_{\mu\nu}$ s such as,

$$F_{\lambda\mu}F^{\mu\rho}F_{\rho}^{\lambda}, \quad (4.41)$$

will yield zero because only F_{12} and F_{21} are non zero. The first thing to notice from the above discussion is that the form-factors themselves are **CP** even and so the **CP** transformation property of $\Pi_{\mu\nu}^5(k)$ is solely dependent on the basis tensors of $\Pi_{\mu\nu}^5(k)$. Secondly, from Eq. (3.3) of chapter 3 it is evident that in the $\mathcal{B} \rightarrow 0$ limit the propagator in a magnetic field transforms into the vacuum propagator and it is shown [55] that the Schwinger propagator can be expanded in a series of powers of the external field magnitude i.e. \mathcal{B} . From this observation it is assumed that calculations based on the

Schwinger propagator are perturbative in the the external field i.e. can be expanded in a series of \mathcal{B} . In the present circumstance the form-factors can also be expanded in a power series of the the even powers of \mathcal{B} where the first term of the series is \mathcal{B}^2 raised to the power zero i.e. a term independent of \mathcal{B} . So if the external magnetic field vanishes the form-factors themselves in general do not vanish. As a result the second rank tensors which multiply the form-factors must tend to zero as the external field is put off because $\Pi_{\mu\nu}^5(k)$ itself vanishes when the external field is absent.

Now all the possible 4-vectors that can be made from the quantities enlisted in Eq. (4.40) are given. They are,

$$\begin{aligned} b_1^\mu &= (Fk)^\mu, \\ b_2^\mu &= (\tilde{F}k)^\mu, \\ b_3^\mu &= k_\parallel^\mu, \\ b_4^\mu &= k_\perp^\mu. \end{aligned} \quad (4.42)$$

The expressions $(Fk)_\mu$, and $(\tilde{F}k)_\mu$ stands for

$$\begin{aligned} (Fk)_\mu &= F_{\mu\nu} k^\nu, \\ (\tilde{F}k)_\mu &= \tilde{F}_{\mu\nu} k^\nu. \end{aligned} \quad (4.43)$$

The set of 4-vectors b_i^μ , $i = 1, 2, 3, 4$ are mutually orthogonal to each other and can serve as the basis vectors to build up the tensor basis of $\Pi_{\mu\nu}^5(k)$.

Next the **CP** transformation properties of these vectors are summarized.

$$\begin{aligned} \text{CP : } b_1^0 &\rightarrow b_1^0, \\ \text{CP : } \mathbf{b}_1 &\rightarrow \mathbf{b}_1. \end{aligned} \quad (4.44)$$

The other three vectors have similar transformation properties as

$$\begin{aligned} \text{CP : } b_{2,3,4}^0 &\rightarrow b_{2,3,4}^0 \\ \text{CP : } \mathbf{b}_{2,3,4} &\rightarrow -\mathbf{b}_{2,3,4}. \end{aligned} \quad (4.45)$$

From Eq. (4.39) we can see that a suitable tensor basis can be built up from vectors b_i^μ where $i = 2, 3, 4$. The **CP** transformation of the axialvector-vector amplitude compels us to disregard b_1^μ as a basis vector.

Now we can list all the possible second rank tensors, having similar **CP** transformation properties to that of $\Pi_{\mu\nu}^5(k)$, made by combining in pairs the mutually orthogonal 4-vectors b_2^μ , b_3^μ , b_4^μ . There are nine of them. For later usage we denote them as follows:

$$\begin{aligned} B_1^{\mu\nu} &= b_2^\mu b_2^\nu = (\tilde{F}k)^\mu (\tilde{F}k)^\nu, & B_2^{\mu\nu} &= b_3^\mu b_3^\nu = k_\parallel^\mu k_\parallel^\nu, & B_3^{\mu\nu} &= b_4^\mu b_4^\nu = k_\perp^\mu k_\perp^\nu, \\ B_4^{\mu\nu} &= b_2^\mu b_3^\nu = (\tilde{F}k)^\mu k_\parallel^\nu, & B_5^{\mu\nu} &= b_3^\mu b_2^\nu = (\tilde{F}k)^\nu k_\parallel^\mu, & B_6^{\mu\nu} &= b_2^\mu b_4^\nu = (\tilde{F}k)^\mu k_\perp^\nu, \\ B_7^{\mu\nu} &= b_4^\mu b_2^\nu = (\tilde{F}k)^\nu k_\perp^\mu, & B_8^{\mu\nu} &= b_3^\mu b_4^\nu = k_\parallel^\mu k_\perp^\nu, & B_9^{\mu\nu} &= b_4^\mu b_3^\nu = k_\parallel^\nu k_\perp^\mu. \end{aligned} \quad (4.46)$$

This basis gives nine second rank mutually orthogonal tensors. The orthogonal condition for the second rank tensors is given by,

$$B_i^{\mu\nu} B_{j\mu\nu} = 0. \quad (4.47)$$

In the above equation $i, j = 1, \dots, 9$ and $i \neq j$. Any second rank tensor containing higher field dependence can be represented by suitable combinations of these tensors.

Out of these nine basis tensors, $B_2^{\mu\nu}$, $B_3^{\mu\nu}$, $B_8^{\mu\nu}$ and $B_9^{\mu\nu}$ do not vanish in the $\mathcal{B} \rightarrow 0$ limit and are therefore redundant. The remaining five second rank tensors qualify successfully as probable building blocks of the axialvector-vector amplitude in the light of **CP** transformation properties. But till now the gauge invariance condition as given in Eq. (4.36) has not been used. Regarding gauge invariance the first thing to note is that $B_1^{\mu\nu}$ can also be written as,

$$\begin{aligned} B_1^{\mu\nu} &= (\tilde{F}k)^\mu (\tilde{F}k)^\nu \\ &= -\mathcal{B}^2(g_{\parallel}^{\mu\nu} k_{\parallel}^2 - k_{\parallel}^\mu k_{\parallel}^\nu), \end{aligned} \quad (4.48)$$

and as a result

$$k_\mu B_1^{\mu\nu} = B_1^{\mu\nu} k_\nu = 0. \quad (4.49)$$

From the above equation it is seen that $B_1^{\mu\nu}$ satisfies the gauge invariance condition for the photon vacuum polarization tensor. From Eq. (4.48) and the expression of the vacuum polarization given in Eq. (4.32) it can be seen that, except a numerical constant which can be absorbed in N_{\parallel} , $B_1^{\mu\nu}$ actually occurs in the expression of the polarization tensor. From these observations it can be inferred that $B_1^{\mu\nu}$ is not a suitable tensor to be in the set of the probable second rank tensors which can act as the basis of axialvector-vector amplitude. The basis tensors of the axialvector-vector amplitude can be made by suitable combinations of the other four second rank tensors listed above, satisfying the **CP** transformation properties as given in Eq. (4.39) and the gauge invariance condition. They are,

$$P_1^{\mu\nu} = (\tilde{F}k)^\nu k_{\parallel}^\mu, \quad (4.50)$$

and

$$P_2^{\mu\nu} = (\tilde{F}k)^\mu k_{\perp}^\nu + (\tilde{F}k)^\nu k_{\perp}^\mu + \frac{k_{\perp}^2}{k_{\parallel}^2} [(\tilde{F}k)^\mu k_{\parallel}^\nu - (\tilde{F}k)^\nu k_{\parallel}^\mu]. \quad (4.51)$$

$P_2^{\mu\nu}$ can be simply written as:

$$P_2^{\mu\nu} = (\tilde{F}k)^\mu k_\perp^\nu + (\tilde{F}k)^\nu k_\perp^\mu + k_\perp^2 \tilde{F}^{\mu\nu}, \quad (4.52)$$

utilizing the identity,

$$\tilde{F}^{\mu\nu} = \frac{1}{k_\parallel^2} \left[(\tilde{F}k)^\mu k_\parallel^\nu - (\tilde{F}k)^\nu k_\parallel^\mu \right]. \quad (4.53)$$

The result as given in the papers by Hari Dass and Raffelt⁹ verifies this choice [26, 28],

$$\Pi_{\mu\nu}^5(k) = \frac{e^3}{(4\pi)^2 m^2} \left[C_\parallel P_{1\mu\nu} + C_\perp P_{2\mu\nu} \right], \quad (4.54)$$

where C_\parallel and C_\perp are the form-factors.

In presence of a magnetized medium

In presence of a magnetized medium the situation complicates. In this analysis we are not going into an in depth study of the tensorial basis as is done in the case where there was no medium. No new techniques are used in this discussion, it is only a continuation of the previous analysis in a new situation.

To start with we again emphasize on the **CP** transformation properties of the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$. Unlike the vacuum case now the theory may not be **CP** invariant. If the background contains more number of electrons than positrons and/or if the electrons momenta are not random but along some specific direction then **CP** is violated by the background. For simplicity here we will discuss only those cases where the background does not break **CP**.

Now the form-factors can be functions of odd powers of the magnetic field as well, as now new scalars as:

$$\begin{aligned} (Fk)u &= (Fk)_\mu u^\mu, \\ (\tilde{F}k)u &= (\tilde{F}k)_\mu u^\mu, \end{aligned} \quad (4.55)$$

are also available. These scalars change sign under **CP** transformation. Some of the form-factors may contain odd powers of the scalars listed above multiplied by equal odd powers of chemical potential of the background charged fermions. These form-factors will

⁹However the metric used by the authors in references mentioned is different from that of ours, and in their calculation the μ vertex is the vector type vertex. Due to this change the sign of $k_\perp^2 \tilde{F}^{\mu\nu}$ appearing in $P_2^{\mu\nu}$ is different from the one they obtained. The metric used in this thesis is specified in section 1.3

then not change sign under a **CP** transformation. So in a magnetized medium there can be two kinds of form-factors, one which does not change sign and the other which does change sign under a **CP** transformation. This is unlike the previous case where there was no medium. So in this case the basis tensors can have different **CP** transformation properties as the form-factors which multiply them can also have different transformation properties. The vector b_1^μ in Eq. (4.42) had a **CP** transformation property which was different from the other three vectors $b_{2,3,4}^\mu$, as a result it was dropped from the set of vectors which could combine in pairs to give a set of second rank tensors having similar **CP** transformation property as that of $\Pi_{\mu\nu}^5(k)$. In the present circumstances b_1^μ is not excluded as in the vacuum because the **CP** transformation property of the basis tensors have changed.

In presence of a medium we can have two different sets of mutually orthogonal vectors. The first set is as supplied in Eq. (4.42). The other set of orthogonal vectors useful in a medium, are

$$\begin{aligned} b_1''^\mu &= (\tilde{F}u)^\mu, \\ b_2''^\mu &= u_\parallel^\mu, \end{aligned} \quad (4.56)$$

In this list we have omitted two vectors. One is $(Fu)^\mu$ and the other one is u_\perp^μ . The reason is that ultimately we are interested in the rest frame of the medium. In the medium rest frame there is no electric field. Also $u_\perp^\mu = 0$ in this frame.

This above set of vectors as given in Eq. (4.56) has similar **CP** transformation properties to those of b_2^μ and b_3^μ . But the vectors given in Eq. (4.42) are not orthogonal to the vectors specified in Eq. (4.56). So although the number of vectors at hand has increased, four from Eq. (4.42) and two from Eq. (4.56), the number of mutually orthogonal vectors have not increased. So this six vectors as such cannot serve as the basis vectors to build up the second rank tensor basis of $\Pi_{\mu\nu}^5(k)$. Only a linear combination of the two sets of vectors can make a orthogonal vector basis. Now we list the set of orthogonal basis vectors which can be made from the two set of vectors, they are:

$$\begin{aligned} b_1'''^\mu &= (Fk)^\mu, \\ b_2'''^\mu &= (\tilde{F}u)^\mu + (\tilde{F}k)^\mu, \\ b_3'''^\mu &= k_\perp^\mu, \\ b_4'''^\mu &= k_\parallel^\mu + u_\parallel^\mu. \end{aligned} \quad (4.57)$$

In a magnetized medium we have these four basis vectors which serves as the building blocks of the axialvector-vector amplitude. The basis tensors in this case will be the

direct product of these basis vectors. There will be sixteen of them but all of them will not be useful. The electromagnetic current conservation condition will reduce the number of admissible basis tensors.

4.2.4 One loop calculation of the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ in a magnetized medium

Since we investigate the case with a uniform background magnetic field, without any loss of generality it can be taken to be in the z -direction. The relevant Feynman diagram of the neutrino electromagnetic vertex to one loop appears in Fig. 4.2. Following that diagram the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ is given as in Eq. (4.20). For calculating $\Pi_{\mu\nu}^5(k)$ in presence of a magnetized medium the propagators appearing in Eq. (4.20) like $S(p)$ and $S(p')$ must be replaced by the appropriate ones suitable, $S_B(p)$ and $S_B(p')$ as introduced in chapter 3, Eq. (3.22).

Similar to the case of $\Pi_{\mu\nu}(k)$ here also the phase factor does not appear because the quantity in question is a one loop 2-point function. From Eq. (3.22), we see that there are two terms in the propagator $S_B(p)$ in a magnetized medium, the vacuum part and the other part which involves background matter distribution. In the trace appearing in the right hand side of Eq. (4.20) there appears a product of two propagators $S_B(p)$ and $S_B(p')$ with other quantities in between. This product will give rise to four quantities of which one term will contain two factors of S_B^V s from the two propagators. It has no importance as far as the background effects are concerned. There will be one term containing two factors of S_B^η s, which will contribute only to the absorptive part of the axialvector-vector amplitude. There will be two terms in the trace each containing a factor of S_B^V and a factor of S_B^η . These terms contribute to the present calculation, as they are important for calculating the effective charge of the neutrinos in a magnetized medium.

With this introduction the relevant expression of $\Pi_{\mu\nu}^5(k)$ can be written as,

$$i\Pi_{\mu\nu}^5(k) = e^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu i S_B^V(p') + \gamma_\mu \gamma_5 i S_B^V(p) \gamma_\nu S_B^\eta(p') \right]. \quad (4.58)$$

Using the form of the fermion propagators in presence of a magnetic field and a magnetized medium, given by Eq. (3.3) and Eq. (3.23) in sections 3.2 and 3.3 of chapter 3 we get

$$\begin{aligned}
i\Pi_{\mu\nu}^5(k) &= -e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\
&\times \int_0^{\infty} ds' e^{\Phi(p',s')} [\text{Tr} [\gamma_\mu \gamma_5 G(p,s) \gamma_\nu G(p',s')] \eta_F(p \cdot u) \\
&\quad + \text{Tr} [\gamma_\mu \gamma_5 G(-p',s') \gamma_\nu G(-p,s)] \eta_F(-p \cdot u)] \\
&= -e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}(p,p',s,s'), \tag{4.59}
\end{aligned}$$

where $R_{\mu\nu}(p,p',s,s')$ contains the traces.

R_{μν} to even and odd orders in magnetic field

We calculate $R_{\mu\nu}(p,p',s,s')$ to even and odd orders in the external magnetic field and call them $R_{\mu\nu}^{(e)}$ and $R_{\mu\nu}^{(o)}$. The reason for doing this is that the two contributions have different properties as far as their dependence on medium is concerned, and the contributions are

$$\begin{aligned}
R_{\mu\nu}^{(e)} &= 4i\eta_-(p \cdot u)\varepsilon_{\mu\nu\alpha\beta} \left[p^{\alpha\parallel} p'^{\beta\parallel} (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right. \\
&\quad + p^{\alpha\parallel} p'^{\beta\perp} \sec^2(e\mathcal{B}s') + p^{\alpha\perp} p'^{\beta\parallel} \sec^2(e\mathcal{B}s) \\
&\quad \left. + p^{\alpha\perp} p'^{\beta\perp} \sec^2(e\mathcal{B}s) \sec^2(e\mathcal{B}s') \right], \tag{4.60}
\end{aligned}$$

and

$$\begin{aligned}
R_{\mu\nu}^{(o)} &= 4i\eta_+(p \cdot u) \left[m^2 \varepsilon_{\mu\nu 12} (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \right. \\
&\quad + \left\{ (g_{\mu\alpha\parallel} p^{\tilde{\alpha}\parallel} p'_{\nu\parallel} - g_{\mu\nu} p'_{\alpha\parallel} p^{\tilde{\alpha}\parallel} + g_{\nu\alpha\parallel} p^{\tilde{\alpha}\parallel} p'_{\mu\parallel}) \right. \\
&\quad + (g_{\mu\alpha\parallel} p^{\tilde{\alpha}\parallel} p'_{\nu\perp} + g_{\nu\alpha\parallel} p^{\tilde{\alpha}\parallel} p'_{\mu\perp}) \sec^2(e\mathcal{B}s') \Big\} \tan(e\mathcal{B}s) \\
&\quad + \left\{ (g_{\mu\alpha\parallel} p'^{\tilde{\alpha}\parallel} p_{\nu\parallel} - g_{\mu\nu} p_{\alpha\parallel} p'^{\tilde{\alpha}\parallel} + g_{\nu\alpha\parallel} p'^{\tilde{\alpha}\parallel} p_{\mu\parallel}) \right. \\
&\quad \left. \left. + (g_{\mu\alpha\parallel} p'^{\tilde{\alpha}\parallel} p_{\nu\perp} + g_{\nu\alpha\parallel} p'^{\tilde{\alpha}\parallel} p_{\mu\perp}) \sec^2(e\mathcal{B}s) \right\} \tan(e\mathcal{B}s') \right]. \tag{4.61}
\end{aligned}$$

Here

$$\eta_+(p \cdot u) = \eta_F(p \cdot u) + \eta_F(-p \cdot u), \tag{4.62}$$

$$\eta_-(p \cdot u) = \eta_F(p \cdot u) - \eta_F(-p \cdot u), \tag{4.63}$$

which contain the information about the distribution functions. Also it should be noted that, in our convention

$$a_\mu b^{\tilde{\mu}\parallel} = a_0 b^3 + a_3 b^0. \tag{4.64}$$

As stated we have split the contributions to $\Pi_{\mu\nu}^5(k)$ to odd and even orders in the external constant magnetic field. The main reason for doing so is the fact that, $\Pi_{\mu\nu}^{5(o)}(k)$ and $\Pi_{\mu\nu}^{5(e)}(k)$, the axialvector-vector amplitude to odd and even powers in $e\mathcal{B}$, have different dependence on the background matter. Pieces proportional to even powers in \mathcal{B} are proportional to $\eta_-(p \cdot u)$, an odd function of the chemical potential. On the other hand pieces proportional to odd powers in \mathcal{B} depend on $\eta_+(p \cdot u)$, and are even in μ and as a result it survives in the limit $\mu \rightarrow 0$.

From Eq. (4.60) we notice that $\Pi_{\mu\nu}^5(k)$ to even orders in the magnetic field satisfies the current conservation condition in both the vertices. In Eq. (4.61) we see that all the terms in the right hand side are symmetric in the μ and ν indices except the first term. This term differentiates between the two vertices in this case and as $\Pi_{\mu\nu}^5(k)$ to odd orders in magnetic field is gauge invariant in the ν vertex we do not get the same condition for the axialvector vertex. If in Eq. (4.61) we put $m = 0$ then all the terms in the right will be symmetric in both the tensor indices, and as a result current conservation condition will hold for both vertices. If the mass of the looping fermion is not zero then from the above analysis we can say that only Eq. (4.36) will hold.

If we concentrate on the rest frame of the medium, then $p \cdot u = p_0$. Thus, the distribution function does not depend on the spatial components of p . In this circumstance we can manipulate under the momentum integral signs, and modify the above equations. The techniques used to convert expressions inside the momentum integrals is briefly discussed in appendix D, while other related calculations showing the derivations of the equations given below are supplied in appendix F. Using the relations discussed in the appendix we can rewrite,

$$\begin{aligned} R_{\mu\nu}^{(e)} &\stackrel{\circ}{=} 4i\eta_-(p_0) \left[\varepsilon_{\mu\nu\alpha\parallel\beta\parallel} p^{\alpha\parallel} p'^{\beta\parallel} (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) + \varepsilon_{\mu\nu\alpha\perp\beta\perp} p^{\alpha\perp} p'^{\beta\perp} \sec^2(e\mathcal{B}s') \right. \\ &\quad \left. + \varepsilon_{\mu\nu\alpha\perp\beta\parallel} p^{\alpha\perp} p'^{\beta\parallel} \sec^2(e\mathcal{B}s) \right], \end{aligned} \quad (4.65)$$

and

$$\begin{aligned} R_{\mu\nu}^{(o)} &\stackrel{\circ}{=} 4i\eta_+(p_0) \left[-\varepsilon_{\mu\nu 12} \left\{ \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_\perp^2 + (k \cdot p)_\parallel (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \right\} \right. \\ &\quad + 2\varepsilon_{\mu 12\alpha\parallel} (p'_{\nu\parallel} p^{\alpha\parallel} \tan(e\mathcal{B}s) + p_{\nu\parallel} p'^{\alpha\parallel} \tan(e\mathcal{B}s')) + g_{\mu\alpha\parallel} k_{\nu\perp} \left\{ p^{\tilde{\alpha}\parallel} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \right. \\ &\quad \left. - k^{\tilde{\alpha}\parallel} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right\} + \{g_{\mu\nu}(p \cdot \tilde{k})_\parallel + g_{\nu\alpha\parallel} p^{\tilde{\alpha}\parallel} k_{\mu\perp}\} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \\ &\quad \left. + g_{\nu\alpha\parallel} k^{\tilde{\alpha}\parallel} p_{\mu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right]. \end{aligned} \quad (4.66)$$

The sign ‘ $\stackrel{\circ}{=}$ ’ means that the expressions on both sides of the above equations, though not necessarily equal algebraically, yield the same momentum integral. To be precise if there are two functions of 4-momenta p , p' and parameters s , s' , as $A(p, p', s, s')$ and $B(p, p', s, s')$ and

$$A(p, p', s, s') \stackrel{\circ}{=} B(p, p', s, s'), \quad (4.67)$$

then in general,

$$A(p, p', s, s') \neq B(p, p', s, s'), \quad (4.68)$$

but

$$\int d^4p A(p, p', s, s') e^{\Phi(p, s) + \Phi(p', s')} = \int d^4p B(p, p', s, s') e^{\Phi(p, s) + \Phi(p', s')}. \quad (4.69)$$

As the momentum integrals appearing in the right hand side of Eq. (4.59) are of the same form as those in Eq. (4.69) we can safely use the right hand sides of Eq. (4.65) and Eq. (4.66) as representatives of $R_{\mu\nu}^{(e)}$ and $R_{\mu\nu}^{(o)}$.

Before going into the next section we comment on the nature of the integral appearing in Eq. (4.59). The first point to make is that from the form of $R_{\mu\nu}^{(e)}$ in Eq. (4.65) we note the axialvector-vector amplitude in a magnetized medium to even orders in the magnetic field is antisymmetric, as it was in a medium without any magnetic field. Contrary to this $R_{\mu\nu}^{(o)}$ does not have any well defined symmetry property.

Secondly, as the integrals are not done explicitly something must be said about possible ultraviolet divergence which may creep up in evaluating them. It is a complicated topic but fortunately the integrals we are dealing with do not have any divergences as we are working in presence of a medium which offers a natural ultraviolet cut-off as the temperature of the system. The penultimate section in this chapter discusses the points about ultraviolet divergences of the two cases we have considered in this chapter.

The integral expression for $\Pi_{\mu\nu}^5(k)$ in our case does not have any singularities. We can now write the full expression of the axialvector-vector amplitude as

$$i\Pi_{\mu\nu}^5(k) = -e^2 \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p, s)} \int_0^{\infty} ds' e^{\Phi(p', s')} [R_{\mu\nu}^{(o)} + R_{\mu\nu}^{(e)}], \quad (4.70)$$

where $R_{\mu\nu}^{(o)}$ and $R_{\mu\nu}^{(e)}$ are given by Eqs.(4.66) and (4.65) in the rest frame of the medium. A thorough proof of the gauge invariance of $\Pi_{\mu\nu}^5(k)$ is given in appendix G.

4.3 Neutrino effective charge in various backgrounds

4.3.1 Definition of effective charge

Intuitively when a neutrino moves inside a thermal medium composed of electrons and positrons, they interact with these background particles. The background electrons and positrons themselves have interaction with the electromagnetic fields, and this fact gives rise to an effective coupling of the neutrinos to the photons. Under these circumstances the neutrinos may acquire an “effective electric charge” through which they interact with the ambient plasma. The proper definition of this charge follows.

The off-shell electromagnetic vertex function Γ_ν is defined in such a way that, for on-shell neutrinos, the $\nu\nu\gamma$ amplitude is given by:

$$\mathcal{M} = -i\bar{u}(q')\Gamma_\nu u(q)A^\nu(k), \quad (4.71)$$

where, k is the photon momentum. Here, $u(q)$ is the the neutrino spinor and A^ν stands for the electromagnetic vector potential. In general Γ_ν would depend on k and the characteristics of the medium. With our effective Lagrangian in Eq. (4.1), the form of Γ_ν is as given in Eq. (4.12). The effective charge of the neutrinos is defined in terms of the vertex function by the following relation [24, 5]:

$$e_{\text{eff}} = \frac{1}{2q_0} \bar{u}(q) \Gamma_0(k_0 = 0, \mathbf{k} \rightarrow 0) u(q). \quad (4.72)$$

For left handed massless Weyl spinors this definition can be rendered into the form:

$$e_{\text{eff}} = \frac{1}{2q_0} \text{Tr} \left[\Gamma_0(k_0 = 0, \mathbf{k} \rightarrow 0) (1 - \gamma^5) \not{q} \right]. \quad (4.73)$$

Both the vector-vector amplitude and the axialvector-vector amplitude contribute to the effective charge of neutrinos. In the following subsections the the effective charge of neutrinos in different backgrounds are discussed.

4.3.2 In vacuum

In vacuum $\Pi_{\mu\nu}(k)$ vanishes in the limit $k_0 = 0$, $\mathbf{k} \rightarrow 0$ and $\Pi_{\mu\nu}^5(k)$ is zero. As a result there can be no effective charge of neutrinos in vacuum.

4.3.3 In a medium

In a medium the axialvector-vector amplitude is of the form $\varepsilon_{\mu\nu\alpha\beta} u^\alpha k^\beta$, and so it does not contribute for the effective electric charge of the neutrinos since for charge calculation

we have to put the index $\nu = 0$. In the rest frame only the time component of the 4-vector u exists, that forces the totally antisymmetric tensor to vanish. In a medium the vector-vector amplitude is same as the vacuum polarization as given in Eq. (4.26) except one point. Eq. (4.26) contains Π_P which is an effect of \mathbf{P} and \mathbf{CP} violation, either by the Lagrangian or by the background medium. Phenomenologically it will come in orders of G_F and so its inclusion will make the effective charge of neutrinos to be higher in orders in the Fermi coupling. So the effect of Π_P is neglected in the effective charge calculations of neutrinos. The longitudinal projector $L_{\mu\nu}$ is not zero in the limit $k_0 = 0, \mathbf{k} \rightarrow 0$ and Π_L is also not zero in the above mentioned limit [24, 5]. This fact is responsible for giving nonzero contribution to the effective charge of neutrinos in a medium. The effective charge of neutrinos in a medium has been calculated previously by many authors [89, 24, 5].

4.3.4 In presence of a background magnetic field

From the Eq. (4.54) we see that the axialvector-vector amplitude in a background magnetic field without any medium does not survive when the momentum of the external photon vanishes. Also from Eq. (4.32) it is evident that $\Pi^{\mu\nu}(k)$ also vanishes when the momentum of the external photon vanishes. As a result there cannot be any effective electric charge of the neutrinos in a constant background magnetic field. Actually this formal statement could have been spoilt by the presence of possible infrared divergence in the loop; i.e to say in C_{\parallel} and C_{\perp} [28] and N_0, N_{\parallel} and N_{\perp} [25]. Since the particle inside the loop is massive so there is no scope of having infrared divergences.

4.3.5 In presence of a magnetized medium

To odd orders in the external magnetic field the expression of $\Pi_{\mu\nu}(k)$ is given by Eq. (4.34) and it shows that the result is dependent on k^β . In the limit when the momentum of the external photon goes to zero $\Pi_{\mu\nu}(k)$ will vanish unless the parametric integrals can produce terms which cancels k^β . In the present circumstance this cancellation does not occur and as a result $\Pi_{\mu\nu}(k)$ vanishes. On the other hand all the components of $\Pi_{\mu\nu}^5(k)$ does not vanish in the specified momentum limit. The relevant components of $\Pi_{\mu\nu}^{5(e)}(k)$ which are required for the effective charge calculation, vanish completely when the external momentum goes to zero. But all the components of $\Pi_{\mu\nu}^{5(o)}(k)$ does not vanish when the external photon momentum goes to zero. As a result this can give rise to an

effective neutrino charge in a magnetized medium. The calculations showing the validity of the above statements appear in appendix F.

Now we concentrate on the zero momentum limit of that part of the axial polarization tensor which is going to contribute for the neutrino effective charge in the rest frame of a magnetized medium.

Effective charge to odd orders in external field

Denoting $\Pi_{\mu\nu}^5(k_0 = 0, \mathbf{k} \rightarrow 0)$ by $\Pi_{\mu\nu}^5$, we obtain

$$\begin{aligned}\Pi_{\mu 0}^5 &= \lim_{k_0=0, \mathbf{k} \rightarrow 0} 4e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \\ &\quad \times \eta_+(p_0) [2p_0^2 - (k \cdot p)_\parallel] \varepsilon_{\mu 012}\end{aligned}\tag{4.74}$$

the other terms turns out to be zero in this limit. Although it looks that the term $(k \cdot p)_\parallel$ will vanish in the limit when $k_0 = 0$ and $\mathbf{k} \rightarrow 0$, but it does not. This is because the parametric integrals yield terms which contain $(k \cdot p)_\parallel$ in the denominator of the momentum integrals and there remains a scope for cancellation. The calculation in appendix F clarifies the point. The above equation shows that, except the exponential functions, the integrand is free of the perpendicular components of momenta. This is a peculiarity of this case that the perpendicular excitations of the loop momenta are only present in the phase like part of the integrals and in effect decouples from the scene once they are integrated out. Its presence is felt only through a linear dependence of the external field \mathcal{B} when the perpendicular components of k vanish. Upon performing the gaussian integration over the perpendicular components and taking the limit $k_\perp \rightarrow 0$, we obtain,

$$\begin{aligned}\Pi_{30}^5 &= \lim_{k_0=0, \mathbf{k} \rightarrow 0} \frac{(4ie^3\mathcal{B})}{4\pi} \int \frac{d^2 p_\parallel}{(2\pi)^2} \int_{-\infty}^{\infty} ds e^{is(p_\parallel^2 - m^2) - \varepsilon|s|} \int_0^{\infty} ds' e^{is'(p'_\parallel^2 - m^2) - \varepsilon|s'|} \\ &\quad \times \eta_+(p_0) [2p_0^2 - (k \cdot p)_\parallel].\end{aligned}\tag{4.75}$$

The details of the calculation for the effective charge is given in appendix F. For a classical gas background the expression of the effective charge, to odd orders in the external magnetic field, is given by,

$$e_{\text{eff}}^\nu = -2\sqrt{2}g_A m \beta G_F \frac{e^2 \mathcal{B}}{\pi^2} \cos \theta \cosh(\beta \mu) K_1(m\beta).\tag{4.76}$$

Here θ is the angle between the neutrino three momentum and the background magnetic field. The superscript ν in e_{eff}^ν denotes that we are calculating the effective charge of

the neutrino. $K_1(m\beta)$ is the modified Bessel function (of the second kind) of order one which sharply falls off as we move away from the origin in the positive direction. As temperature tends to zero, $K_1(m\beta)$ grows as $e^{-m\beta}$, as a result from Eq. (4.76) it is seen that the effective charge vanishes. In [33] a more general treatment of the effective charge, relinquishing the assumption of a classical gas, has been presented.

Before ending this section a general discussion can be made on the form-factors of $\Pi_{\mu\nu}^5(k)$ in the zero momentum limit. In a background magnetic field the form-factors can be functions of scalars containing the magnetic field. The scalars can be of the following form:

$$k^\mu F_{\mu\nu} F^{\nu\lambda} k_\lambda \quad \text{and} \quad F_{\mu\nu} F^{\mu\nu}, \quad (4.77)$$

or

$$(\tilde{F}u)^\mu (\tilde{F}u)_\mu \quad \text{and} \quad (\tilde{F}u)^\mu k_\mu. \quad (4.78)$$

The scalars listed above are the most elementary ones, there can be many more possible scalars where more number of k^μ s, $F^{\mu\nu}$ s and u_μ s are mutually fully contracted. The thing which must be noted is when k tends to zero only terms that can survive in the form-factors must be even functions of \mathcal{B} .

Of all possible tensorial structures for the axialvector-vector amplitude in a magnetized plasma, there exists one which is independent of the external momentum k given by,

$$\tilde{F}_{\mu\alpha} u^\alpha u^\parallel. \quad (4.79)$$

It is seen from the above expression, which is odd in the external field, that it survives in the zero external momentum limit in the rest frame of the medium. We have earlier noted that the form-factors which exist in the rest frame of the medium and in the zero momentum limit are even in powers of the external field. This tells us directly that the axial polarization tensor must be odd in the external field in the zero external momentum limit, a result which is verified by actual calculations. Some similar calculations regarding the axialvector-vector coupling is also done by Konar and Das [90].

4.4 A short note on divergences in a background magnetic field

While calculating closed loops in a Feynman diagram in vacuum we encounter ultraviolet divergences in many places. For renormalizable quantum field theories there are

standard methods of handling this divergences. First the theory is regulated and then some of the bare parameters in the theory are rescaled to give a meaningful theory. There are various regularization schemes as the cut-off regularization, the Pauli-Villars regularization and the most prominently used dimensional regularization. Not going into the full details of the renormalization programme in vacuum, which has been studied extensively, here we focus on the methods by which ultraviolet divergences are handled in those calculations where we have a background magnetic field.

The first thing to note here is that magnetic fields do not bring in any new kind of divergences, they only modify the properties of the divergences present in the vacuum structure of the theory. Vacuum polarization of photon contains ultraviolet divergence and consequently the divergence remains when we calculate it in a background magnetic field. The expression of $\Pi_{\mu\nu}(k)$ calculated in a magnetic field includes the $\mathcal{B} = 0$ vacuum result implicitly, which we denote by $\lim_{\mathcal{B} \rightarrow 0} \Pi_{\mu\nu}(k)$. The divergent part in $\Pi_{\mu\nu}(k)$ calculated in a background magnetic field is actually present in $\lim_{\mathcal{B} \rightarrow 0} \Pi_{\mu\nu}(k)$. The usual procedure is to fix the divergent part in $\lim_{\mathcal{B} \rightarrow 0} \Pi_{\mu\nu}(k)$ and subtract it from the the final expression of $\Pi_{\mu\nu}(k)$ calculated in presence of a magnetic field, to get rid of the divergence.

The ultraviolet nature of $\Pi_{\mu\nu}^5(k)$ is interesting. In vacuum we have seen from subsection 4.2.3 that $\Pi_{\mu\nu}^5(k)$ is zero and so it cannot have any divergences when we calculate it in presence of a background magnetic field. In this connection it can be said that in absence of the medium but in presence of the background magnetic field another divergent structure could arise, the Adler anomaly [91], due to the presence of the axial-vector vertex in the Feynman diagram of $\Pi_{\mu\nu}^5(k)$. A brief discussion of this topic is given in the work by Raffelt and Ioannisian [28].

As stated previously a medium affects the ultraviolet nature of the calculations. The finite temperature of the system automatically act as an ultraviolet cut-off of the system and so momentum integrals cannot run to infinity to damage the logical consistency of the theory. As a result the calculations in a magnetized medium are also expected to be free from ultraviolet divergences.

4.5 Conclusion

This chapter was about the electromagnetic interactions of neutrinos in a magnetized medium. In the introduction various decay processes as the photon splitting into two

neutrinos, the Cherenkov process and radiative decays of heavy neutrinos were discussed. Section 4.2 deals with neutrino-photon scattering and the electromagnetic vertex of neutrinos. The topic of neutrino-photon scattering, which is highly suppressed in the standard model, is briefly discussed in subsection 4.2.1. The interesting thing about this scattering process is that in presence of an external magnetic field the neutrino-photon cross-section increases which can be an important ingredient for astrophysics.

In the standard model of particle physics neutrinos do not interact with photons in the tree level so the neutrino electromagnetic vertex is an effective one mediated by the charged fermions in the loop. Section 4.2 deals with the two important second rank tensors, $\Pi_{\mu\nu}(k)$ and $\Pi_{\mu\nu}^5(k)$, which are the building blocks of the electromagnetic vertex function of the neutrinos. The structure of $\Pi_{\mu\nu}(k)$ in various possible backgrounds has been discussed in subsection 4.2.2. Here for the first time methods of statistical field theory is used to find out the form of $\Pi_{\mu\nu}(k)$, to odd orders in the external field, in a magnetized medium. Subsection 4.2.3 of the present chapter deals with the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$ in a magnetized medium and its possible tensor structures in various possible backgrounds. Section 4.3 deals with the issue of the effective charge of the neutrino in a magnetized medium. The formula of the effective charge calculated shows that it depends upon the angle between the neutrino 3-momentum and the external magnetic field directions. Also the effective charge vanishes as temperature tends to zero.

As no discourse in quantum field theory can be complete without solving the thorny issue of divergences popping out from the loops, in section 4.4 this issue is discussed. It is pointed out that the temperature of the medium can act as an ultraviolet regulator for the various processes which we have discussed and we can safely assume that the calculations are free of ultraviolet divergences.

CHAPTER 5

Conclusion

It is now an established fact that magnetic fields of various magnitudes and various shapes pervade our universe. They are present in the intergalactic medium, in the galactic region and also in planetary and stellar atmospheres. Order of magnitude estimates of these fields have been presented in chapter 1.

Two different ways have been followed in this thesis to incorporate the magnetic field effects. The first way is straight forward and involves the solution of the Dirac equation in presence of a classical uniform background magnetic field. In presence of a magnetic field the Dirac equation yields an exact solution. The exact wave functions in a uniform magnetic field can be used to produce a consistent quantum field theory and various scattering cross-sections involving charged fermions can be calculated, as elaborately discussed in the chapter 2. Magnetic fields found in astrophysical objects like neutron stars, active galactic nuclei vary both in magnitude and direction but the typical length scale D in which this variation occurs is much much higher than the Compton wavelengths of the elementary particles and as a result this variation can be neglected in a quantum field theoretical calculation¹⁰.

Magnetic fields also enter into the elementary particle regime indirectly through the effect of virtual charged particles. To evaluate quantities like neutrino self-energy and vacuum polarization of photon intermediate virtual charged particles like electrons, W-bozons are required whose propagators are modified in a magnetic field. Although ultimately the intermediate particles are integrated out, the magnetic field effects remains frozen in and affect the final results. In chapters 3 and 4 we have encountered some of the calculations involving the Schwinger propagator and seen how magnetic fields creep in to modify the physical properties of those particles which does not have any interaction with a magnetic field normally.

¹⁰For neutron stars $D \sim 10\text{Km}$.

Historically the URCA processes, as specified in section 1.2, were the first to be studied in an external field. Closely related with the URCA processes is the inverse beta decay process. In chapter 2 the inverse beta decay cross-section of arbitrarily polarized neutrons in an external magnetic field is calculated. The magnitude of the external magnetic field assumed is much less than m_p^2/e and so the magnetic field modification of the proton wave function is neglected. Using the solutions of the Dirac equation for electrons in a background of a magnetic field a quantum field theoretical formalism is built up to tackle various scattering processes consistently. From Eq. (2.76) of chapter 2 it is seen that the inverse beta decay scattering cross-section in a background magnetic field has a smooth $\mathcal{B} \rightarrow 0$ limit. In Eq. (2.72) the cross-section of the inverse beta decay process is given and is seen to be dependent on the angle between the incoming neutrino 3-momentum and the magnetic field vector. From Fig. 2.1 it is apparent that the cross-section of the inverse beta decay in an external magnetic field becomes considerably enhanced compared to its corresponding value in vacuum when the magnetic field magnitude is greater than \mathcal{B}_e , the critical field defined in Eq. (1.7). Moreover when the initial neutrinos are taken to be monochromatic Fig. 2.1 shows that the cross-section of the inverse beta decay process is plagued with spikes which goes all the way to infinity. This disease is cured and the spikes smear out, as shown in Fig. 2.2, when a flat probability distribution of initial neutrino energy is considered. Using the fact that the beta decay cross-section is sensitive to the angle between the neutrino 3-momentum and the magnetic field direction attempt has been made to explain the high velocities of pulsars, of the order of $450 \pm 90 \text{ Km s}^{-1}$ in section 2.5.

The charged fermion propagator in a background magnetic field as derived by Schwinger is given in chapter 3, subsection 3.2.2. The propagator is defined as an integral over a parameter called ‘proper time’. Using form-factor analysis the general structure of the neutrino self-energy in a magnetized medium has been found out in section 3.4. From Eq. (3.39) it is seen that in a magnetized medium the neutrino self-energy becomes sensitive to the angle between the neutrino propagation direction and the magnetic field direction. It is also sensitive to the nature of the background i.e., whether the background is made up of electrons or muons etc., as is evident from Eq. (3.42). Both of these facts affect the resonant level crossing condition in context of neutrino oscillations discussed in section 3.5.

The topic of electromagnetic interaction of neutrinos is discussed in chapter 4. All the calculations assumes an effective 4-fermi vertex. There are various processes which are either forbidden or highly restrained in vacuum, like a photon decaying into a neutrino-antineutrino pair, Cherenkov radiation of neutrinos, neutrino-photon scattering, which can become possible in presence of a background magnetic field. These processes, briefly discussed in sections 4.1 and 4.2, opens up new channels for energy emission from newly born neutron stars and so are astrophysically important. The neutrino electromagnetic vertex function Γ_μ can be written in terms of two second rank tensors $\Pi_{\mu\nu}(k)$ and $\Pi_{\mu\nu}^5(k)$ as mentioned in chapter 1 and explicitly shown in Eq. (4.12). $\Pi_{\mu\nu}(k)$ is exactly the vacuum polarization of the photon and its form in a thermal medium and in presence of a magnetized medium is presented in subsection 4.2.2. The general form of the axialvector-vector amplitude $\Pi_{\mu\nu}^5(k)$, as defined in Eq. (4.20), is calculated in a magnetized medium in section 4.2.3. A general tensor analysis based on the discrete symmetries shows that $\Pi_{\mu\nu}^5(k)$ must vanish in vacuum. A similar analysis gives the tensor structure of $\Pi_{\mu\nu}^5(k)$ in a thermal medium. Based upon **CP** transformation property of $\Pi_{\mu\nu}^5(k)$ the tensor basis of the axialvector-vector amplitude is analyzed in a background magnetic field. Some comments on the tensor basis of $\Pi_{\mu\nu}^5(k)$ in a magnetized medium is supplied at the end of subsection 4.2.3. Subsequently $\Pi_{\mu\nu}^5(k)$ has been calculated in a specific limit i.e., when $k_0 = 0$ and $\mathbf{k} \rightarrow 0$, which is related to the effective charge of the neutrinos. In this limit it is observed that $\Pi_{\mu\nu}^5(k)$ does exist to odd orders in the external magnetic field but to even orders it does not. It is seen that when $k_0 = 0$ and $\mathbf{k} \rightarrow 0$, $\Pi_{\mu\nu}^5(k)$ becomes linear in the external field. Using a classical gas approximation the expression of the effective charge to odd orders in the external field has been written down. The formula of the effective charge calculated shows that it depends upon the angle between the neutrino 3-momentum and the external magnetic field directions. Also the effective charge vanishes as temperature tends to zero.

To conclude it must be said that most of the effects discussed in this thesis depends upon large magnetic fields, where $\mathcal{B} > \mathcal{B}_e$. These fields are obtained only at astronomical distances, presumably in neutron stars and magnetars. For such distant objects, observational data are not clean enough to resolve the effects of the magnetic field. Perhaps more accurate measurements of indirect evidences like the velocities of the pulsars or rate of cooling of newly born neutron stars can shine some light on the calculations done in this thesis in near future.

APPENDIX A

Spin Sum in a uniform background magnetic field

From Eq. (2.24) the spin sum $\sum_s U_s(y, n, \mathbf{p}_y) \bar{U}_s(y_*, n, \mathbf{p}_y)$ can be written as:

$$\sum_s U_s(y, n, \mathbf{p}_y) \bar{U}_s(y_*, n, \mathbf{p}_y) = \frac{1}{E_n + m} \sum_{i,j=n-1}^n I_i(\xi) I_j(\xi_*) T_{i,j} \quad (\text{A.1})$$

where the $T_{i,j}$ s are 4×4 matrices. The $T_{i,j}$ matrices are obtained by performing the matrix multiplication in the left hand side of the above equation.

Using the dispersion relation $E_n^2 = p_z^2 + m^2 + 2ne\mathcal{B}$, $T_{n,n}$ can be written as,

$$T_{n,n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (E_n + m) & 0 & p_z \\ 0 & 0 & 0 & 0 \\ 0 & -p_z & 0 & -(E_n - m) \end{pmatrix}. \quad (\text{A.2})$$

In the 2×2 notation the above matrix can be written as,

$$T_{n,n} = E_n \begin{pmatrix} \frac{1}{2}(1 - \sigma_3) & 0 \\ 0 & -\frac{1}{2}(1 - \sigma_3) \end{pmatrix} + p_z \begin{pmatrix} 0 & \frac{1}{2}(1 - \sigma_3) \\ -\frac{1}{2}(1 - \sigma_3) & 0 \end{pmatrix} + m \begin{pmatrix} \frac{1}{2}(1 - \sigma_3) & 0 \\ 0 & \frac{1}{2}(1 - \sigma_3) \end{pmatrix}, \quad (\text{A.3})$$

where σ_3 is the third Pauli matrix. In the 4×4 notation Eq. (A.3) can be written as,

$$\begin{aligned} T_{n,n} &= \frac{1}{2}[m(1 - \sigma_z) + E_n(\gamma^0 + \gamma^5\gamma^3) - p_z(\gamma^5\gamma^0 + \gamma^3)], \\ &= \frac{1}{2}[m(1 - \sigma_z) + \not{p}_{\parallel} + \tilde{\not{p}}_{\parallel}\gamma_5], \end{aligned} \quad (\text{A.4})$$

where $\sigma_z = i\gamma^1\gamma^2$.

In a similar way $T_{n-1,n-1}$ can be written as:

$$T_{n-1,n-1} = \begin{pmatrix} (E_n + m) & 0 & -p_z & 0 \\ 0 & 0 & 0 & 0 \\ p_z & 0 & -(E_n - m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

In the 2×2 notation the above equation looks like,

$$T_{n-1,n-1} = E_n \begin{pmatrix} \frac{1}{2}(1 + \sigma_3) & 0 \\ 0 & -\frac{1}{2}(1 + \sigma_3) \end{pmatrix} + p_z \begin{pmatrix} 0 & -\frac{1}{2}(1 + \sigma_3) \\ \frac{1}{2}(1 + \sigma_3) & 0 \end{pmatrix} + m \begin{pmatrix} \frac{1}{2}(1 + \sigma_3) & 0 \\ 0 & \frac{1}{2}(1 + \sigma_3) \end{pmatrix}. \quad (\text{A.6})$$

In the 4×4 notation the above equation becomes,

$$\begin{aligned} T_{n-1,n-1} &= \frac{1}{2}[m(1 + \sigma_z) + E_n(\gamma^0 - \gamma^5\gamma^3) + p_z(\gamma^5\gamma^0 - \gamma^3)], \\ &= \frac{1}{2}[m(1 + \sigma_z) + \not{p}_\parallel - \not{\tilde{p}}_\parallel\gamma_5]. \end{aligned} \quad (\text{A.7})$$

From the matrix multiplication in the left hand side of Eq. (A.1) it can be seen that $T_{n-1,n}$ is given as,

$$T_{n-1,n} = \sqrt{2ne\mathcal{B}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.8})$$

In the 2×2 notation the above equation looks like,

$$T_{n-1,n} = \sqrt{2ne\mathcal{B}} \begin{pmatrix} 0 & \frac{1}{2}(\sigma_1 + i\sigma_2) \\ -\frac{1}{2}(\sigma_1 + i\sigma_2) & 0 \end{pmatrix}. \quad (\text{A.9})$$

Here σ_1 and σ_2 are the first two Pauli matrices. When converted back to the 4×4 notation the above equation becomes,

$$T_{n-1,n} = -\frac{1}{2}\sqrt{2ne\mathcal{B}}(\gamma_1 + i\gamma_2). \quad (\text{A.10})$$

Similarly $T_{n,n-1}$ is given by,

$$T_{n,n-1} = \sqrt{2ne\mathcal{B}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.11})$$

In the 2×2 notation the above equation looks like,

$$T_{n,n-1} = \sqrt{2ne\mathcal{B}} \begin{pmatrix} 0 & \frac{1}{2}(\sigma_1 - i\sigma_2) \\ -\frac{1}{2}(\sigma_1 - i\sigma_2) & 0 \end{pmatrix}, \quad (\text{A.12})$$

which when converted back to the 4×4 notation becomes,

$$T_{n-1,n} = -\frac{1}{2}\sqrt{2ne\mathcal{B}}(\gamma_1 - i\gamma_2). \quad (\text{A.13})$$

Supplying the values of $T_{i,j}$ s from Eq. (A.4), Eq. (A.7), Eq. (A.10) and Eq. (A.13) to Eq. (A.1) we get the result given in Eq. (2.27). A similar procedure is followed to obtain Eq. (2.28).

APPENDIX B

Phase factor of the Schwinger propagator

B.1 Loops of charged particles

As we have discussed previously in chapter 3, the phase factor in the Schwinger propagator appears because the fermion propagator attaches two points with different gauge transformation properties. The Schwinger mechanism is best suited for processes where we do not have external charged particles and all the charged particles are inside the loop. For example one of the figures that contribute to the neutrino self-energy and another one which contributes for two neutrino three photon scattering as shown in Fig. 1.1 are perfect examples for these kind of diagrams. One important fact that can be noticed from the form of the propagators of charged fermions or gauge bosons [64] is that the form of the phase factor¹¹ of the different propagators are the same as given in Eq. (3.10). To understand its importance we take concrete examples.

First we take a one loop two point function containing virtual charged fermion internal lines. The one loop photon vacuum polarization diagram as shown in Fig. B.1 is a good example. The electron propagators connect points P and Q in space time. If we are interested in finding out the overall phase factor accompanying the vacuum polarization tensor then we will have to use the Schwinger propagator for the electrons. The contribution from the phase factors $\kappa(Q, P)$ and $\kappa(P, Q)$ which we denote as $\Psi(P, Q)$ will be according to Eq. (3.6) and Eq. (3.10)

$$\begin{aligned}\Psi(P, Q) &= \kappa(Q, P)\kappa(P, Q) \\ &= \exp\left\{ie\frac{1}{2}[P^\mu F_{\mu\nu}Q^\nu + Q^\mu F_{\mu\nu}P^\nu]\right\},\end{aligned}\tag{B.1}$$

¹¹In the reference cited the authors did not explicitly take $\lambda(\xi)$ as they were not interested in the general properties of the phase factor.

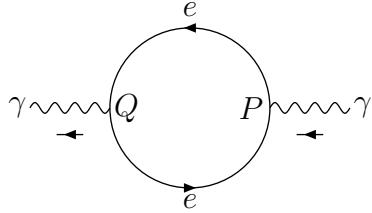


Figure B.1: One loop two point function containing virtual charged fermion internal lines, Vacuum polarization diagram of photon.

which reduces to unity because of the antisymmetry of $F_{\mu\nu}$. From Eq. (B.1) it is seen that the phase factor's contribution in the one loop calculation is trivial and obviously gauge invariant. In any kind of a loop where we have two charged particles the phase factor contribution is unimportant as all the charged particles have the same form of phase factor with the propagators.

Next we take a loop with three charged particle propagators connecting three space-time points P, Q, R . We have shown that the phase factor between two points do not depend upon the path which joins them. Consequently to find the contribution of the phase factors in the calculation for the loop we join the three points by straight lines as shown in Fig. B.2. The overall phase contribution can then be calculated using Eq. (3.6) and Eq. (3.10) and is given by

$$\begin{aligned}\Psi(P, Q, R) &= \kappa(Q, P)\kappa(R, Q)\kappa(P, R) \\ &= \exp\left\{ie\frac{1}{2}[P^\mu F_{\mu\nu}Q^\nu + Q^\mu F_{\mu\nu}R^\nu + R^\mu F_{\mu\nu}P^\nu]\right\}.\end{aligned}\quad (\text{B.2})$$

As all the phase factors are of the same form the first point to notice is that the contribution from the function $\lambda(\xi)$ cancel out in the overall factor, showing that the contribution is explicitly gauge invariant.

The next point which requires to be discussed is about the path independence of the phase factor. From subsection 3.2.2 we know that $\kappa(Q, P)$ is independent of the path which joins them, but here the path independence of $\kappa(Q, P)$ does not imply

$$\Psi(P, Q, R) = \kappa(Q, P)\kappa(R, Q)\kappa(P, R) = 1, \quad (\text{B.3})$$

or from Eq. (3.10)

$$I(P, P) = I(Q, P) + I(R, Q) + I(P, R) = 0. \quad (\text{B.4})$$

Instead of the above expectation we get a finite contribution from Eq. (B.2). As $I(Q, P)$ consists of the product of the two end points instead of their difference, the different phase factors from the different paths connecting two intermediate points of the loop when multiplied does not reduce to unity.

It can be shown in general that if we include loops where we have more than three charged particle propagators joining more than three points in space time then also the phase factor is going to contribute. To understand where we can actually neglect the phase factor and where not we have to calculate the flux of the uniform magnetic field attached to the area of the loop.

B.2 The flux rule

The phase factor contribution to the one loop calculations of various diagrams as shown in Fig. B.1 and Fig. B.2 show that they are explicitly gauge invariant. In the case of Fig. B.1 it was shown that it contributes nothing for the phase factor. The contribution from Fig. B.2 can be understood in another way also. As the magnetic field is directed along the z -direction the component of area enclosed by the looping particles along the $x - y$ plane is only active for its flux. From the path independence of κ we know that $\Psi(P, Q, R)$ can always be calculated from the triangular path connecting the three points

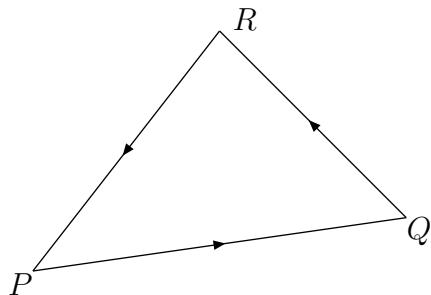


Figure B.2: A loop consisting of three space-time points joined by charged particle propagators. The arrow heads indicate the direction of charge flow.

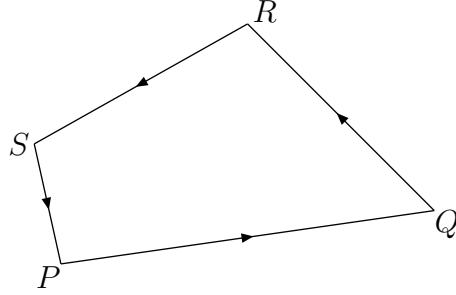


Figure B.3: A loop consisting of four space-time points joined by charged particle propagators. The arrow heads indicate the direction of charge flow.

as shown in Fig. B.2 and so the flux attached with this loop must be the magnitude of the magnetic field \mathcal{B} , times the area of this triangle in the $x - y$ plane denoted by $\Delta_{\perp}(P, Q, R)$. Calculating the flux we get

$$\begin{aligned}\mathcal{F}(P, Q, R) &= \mathcal{B} \Delta_{\perp}(P, Q, R) \\ &= \frac{1}{2} \mathcal{B} [(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{Q})]_z,\end{aligned}\quad (\text{B.5})$$

where $[\mathbf{M} \times \mathbf{N}]_z$ represents z -component of the antisymmetric cross-product between two arbitrary 3-vectors \mathbf{M} and \mathbf{N} . Eq. (B.5) can also be written as

$$\mathcal{F}(P, Q, R) = \frac{1}{2} (Q - P)^{\mu} F_{\mu\nu} (R - Q)^{\nu}, \quad (\text{B.6})$$

and in this form we see from Eq. (B.2) that it can be written as

$$\Psi(P, Q, R) = \exp \{ie\mathcal{F}(P, Q, R)\}. \quad (\text{B.7})$$

Next we consider a loop containing four charged particle propagators. The four photon interaction in QED is a process where this loop occurs naturally. Here the internal virtual particles may be electrons. In coordinate space the four points where the photons attach to the loop may be taken as P, Q, R and S . As far as the contribution from the overall phase is concerned we can join them by straight lines forming a general quadrilateral as shown in Fig. B.3. Using the same technique for finding the overall phase as discussed in section B.1 we find here

$$\begin{aligned}\Psi(P, Q, R, S) &= \kappa(Q, P)\kappa(R, Q)\kappa(S, R)\kappa(P, S) \\ &= \exp \left\{ ie \frac{1}{2} [P^{\mu} F_{\mu\nu} Q^{\nu} + Q^{\mu} F_{\mu\nu} R^{\nu} + R^{\mu} F_{\mu\nu} S^{\nu} + S^{\mu} F_{\mu\nu} P^{\nu}] \right\},\end{aligned}\quad (\text{B.8})$$

which can also be written as

$$\Psi(P, Q, R, S) = \exp \{ie\mathcal{F}(P, Q, R, S)\} , \quad (\text{B.9})$$

where

$$\mathcal{F}(P, Q, R, S) = \frac{1}{2} [(Q - P)^\mu F_{\mu\nu} (R - Q)^\nu + (S - R)^\mu F_{\mu\nu} (P - S)^\nu] . \quad (\text{B.10})$$

Working similarly as was done in the case of Eq. (B.6) it can be shown that the above flux is just the sum of the areas of the triangles PQR and SRP along the $x - y$ plane times the magnitude of the magnetic field.

From Eq. (B.7) and Eq. (B.9) we can generalize that the overall phase depends on the flux of the magnetic field attached with the area of the loop. Given an arbitrary diagram we can initially calculate this flux to find out the overall phase. Also this connection with magnetic flux explains another important fact that this phase is absent for a loop consisting of just two points. If we join this two points with straight lines then in no way will those straight lines enclose any area and as a result no resultant flux will emerge.

The Schwinger propagator as given in Eq. (3.2) is specific for the case of those gauges which gives a background magnetic field. In the general derivation by Schwinger [23] the form of the fermionic propagator was derived for general constant gauge fields. If we choose a pure gauge field which does not give rise to any electric or magnetic field as

$$A_\mu(\xi) = K\delta_\mu^\alpha \xi_\alpha , \quad (\text{B.11})$$

where K is a constant, then also we can find the Schwinger propagator and it differs from the vacuum propagator only by the phase factor. Denoting this propagator in presence of a pure gauge as given in Eq. (B.11) as $iS_A(x, x')$ we have

$$iS_A(x, x') = \exp(ieI(x, x')) \int \frac{d^4 p}{2\pi^4} \frac{i(p + m)}{(p^2 - m^2)} , \quad (\text{B.12})$$

where $I(x, x')$ is given by Eq. (3.13). From generalized Stokes theorem, which is

$$\oint d\xi^\mu A_\mu(\xi) = \int d\sigma^{\mu\nu} F_{\mu\nu}(\xi) , \quad (\text{B.13})$$

where $d\sigma^{\mu\nu}$ is the infinitesimal surface area in the $\xi_\mu - \xi_\nu$ plane and $F_{\mu\nu}$ is the field strength tensor, we can say that $I(x, x')$ is path independent as here $F_{\mu\nu} = 0$. The path

independence of $I(x, x')$ allows us to join the two points by a straight line and proceeding similarly as done in chapter 3 while discussing about the phase factor, we get

$$\begin{aligned} I(x, x') &= \int_{x'}^x d\xi^\mu A_\mu(\xi) \\ &= K(x^\alpha x_\alpha - x'^\alpha x'_\alpha). \end{aligned} \quad (\text{B.14})$$

The above result implies that the overall phase factor for diagrams like those in Fig. B.2 and Fig. B.3 will always be unity. As a result rates of those processes will remain similar to the vacuum results.

APPENDIX C

A short note on ‘Real Time Formulation’

C.1 A brief introduction to thermal propagators

There are more than one way in which we can proceed in real time formalism, out of them here we take the canonical approach[56] as is prevalent in the zero temperature case. The main purpose of this section is to give a glimpse of the real time formalism, to show how a genuinely temperature dependent term and a temperature independent term appears in the propagators. Also some of the relevant formulas will be cited in this section.

In this section, the essence of the real time formulation is presented in a heuristic manner using the real scalar fields. Suppose that we have a Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2, \quad (\text{C.1})$$

which describes a real scalar field. The Fourier mode expansion of the free field is given as,

$$\phi(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2E_p}} [a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}], \quad (\text{C.2})$$

where the $a(p)$ and $a^\dagger(p)$ are the annihilation and the creation operators of the bosons with momentum p . Formally the propagator is represented as

$$i\Delta_F(x - y) = \langle 0 | T\phi(x)\phi(y) | 0 \rangle, \quad (\text{C.3})$$

where T is the time ordered product defined as

$$T\phi(x)\phi(y) = \Theta(x_0 - y_0)\phi(x)\phi(y) + \Theta(y_0 - x_0)\phi(y)\phi(x). \quad (\text{C.4})$$

For obtaining the expression of the propagator from Eq. (C.3) we use the fact that $\langle 0 | a^\dagger(p)a(p) | 0 \rangle = 0$, always due to the basic definition of the operators $a(p)$ and $a^\dagger(p)$, and get

$$\Delta_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}. \quad (\text{C.5})$$

In a medium there are real particles and so the expectation value of the number operator $a^\dagger a$ is not zero $\langle a^\dagger(p)a(p) \rangle_\beta \neq 0$. So now we take

$$\langle a^\dagger(p)a(p') \rangle_\beta = \delta^3(\mathbf{p} - \mathbf{p}') f_B(p), \quad (\text{C.6})$$

where

$$f_B(p) = \frac{1}{e^{\beta p \cdot u} - 1},$$

is the Bose-Einstein distribution function. Here u is the four-velocity of the centre of mass of the heat bath. By the commutation relations naturally we have

$$\langle a(p)a^\dagger(p') \rangle_\beta = \delta^3(\mathbf{p} - \mathbf{p}') [f_B(p) + 1]. \quad (\text{C.7})$$

In a thermal medium the following relations

$$\langle a^\dagger(p)a^\dagger(p') \rangle_\beta = 0 \text{ and } \langle a(p)a(p') \rangle_\beta = 0,$$

remain the same as those in vacuum. Utilizing these facts and doing the calculation as is done in the zero temperature case we find the expression of the thermal propagator as

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\epsilon} - 2\pi i \delta(p^2 - m^2) \frac{1}{e^{\beta p \cdot u} - 1}, \quad (\text{C.8})$$

which has a genuine temperature dependent term as the second term in the right hand side. Here the temperature dependent term comes in with a delta-function showing this propagator corresponds to exchange of real particles unlike the exchange of virtual particles present in the first term. The expression becomes realistic because the thermal bath always has some real particles, so in any process these particles can be exchanged. Eq. (C.8) is really not the whole story, there are other propagators also accompanying it. In the next section some technical details is given which will elucidate the complex structure of this formalism.

C.2 A bit of formal theory

C.2.1 The n-point Green's function in vacuum

The Fourier expansion of the scalar field is given in Eq. (C.2). When we are talking about interacting field theory the free field expansion as given in Eq. (C.2) is often called $\phi_{in}(x)$. To do interacting field theory we can always find an operator U which transforms the incoming free fields ϕ_{in} to the interacting Heisenberg fields $\phi(x)$, and the transformation is given by

$$\phi(x) = U^{-1}\phi_{in}(x)U, \quad t = x^0. \quad (\text{C.9})$$

For the above transformation to hold, we must have

$$U(t) = T \exp \left[i \int_{-\infty}^t \int d^3x \mathcal{L}_{int}[\phi_{in}(x)] \right], \quad (\text{C.10})$$

which is the evolution operator. It must satisfy the following property,

$$\lim_{t \rightarrow -\infty} U(t) = 1. \quad (\text{C.11})$$

Generally an n-point Green's function in the interacting theory is defined as

$$G(x_1, \cdot, \cdot, \cdot, x_n) = \langle 0 | T\{\phi(x_1)\phi(x_2) \cdots \phi(x_n)\} | 0 \rangle. \quad (\text{C.12})$$

Introducing a very large time τ such that

$$-\tau < t_1, t_2, \cdot, \cdot, \cdot, t_n < \tau \quad (\text{C.13})$$

and choosing the convention

$$U(\tau, -\tau) = U(\tau)U^{-1}(-\tau), \quad (\text{C.14})$$

we can manipulate the expression inside the time ordering. Using Eq. (C.9),

$$\begin{aligned} & T\{\phi(x_1)\phi(x_2) \cdots \phi(x_n)\} \\ &= T\{U^{-1}(t_1)\phi_{in}(x_1)U(t_1)U^{-1}(t_2)\phi_{in}(x_2)U(t_2) \cdots U^{-1}(t_n)\phi(x_n)U(t_n)\} \\ &= U^{-1}(\tau) \left[T U(\tau) U^{-1}(t_1)\phi_{in}(x_1)U(t_1) \cdots U^{-1}(t_n)\phi_{in}(x_n)U(t_n) U^{-1}(-\tau) \right] U(-\tau) \\ &= U^{-1}(\tau) \left[T \{\phi_{in}(x_1)\phi_{in}(x_2) \cdots \phi_{in}(x_n)\} U^{-1}(\tau, -\tau) \right] U(-\tau). \end{aligned} \quad (\text{C.15})$$

Using the above equation we can write

$$G(x_1, \cdot, \cdot, \cdot, x_n) = \langle 0 | U^{-1}(\tau) [T\{\phi_{in}(x_1) \cdots \phi_{in}(x_n)\} U(\tau, -\tau)] U(-\tau) | 0 \rangle. \quad (\text{C.16})$$

In vacuum

$$\lim_{\tau \rightarrow \infty} U(\tau)|0\rangle = e^{i\delta}|0\rangle,$$

$$\lim_{\tau \rightarrow -\infty} U(-\tau)|0\rangle = |0\rangle,$$

where δ is some real number. To take account of the phase the Green's function in vacuum is defined as

$$G^{(0)}(x_1, \cdot, \cdot, \cdot, x_n) = \frac{\langle 0 | [T\{\phi_{in}(x_1) \cdots \phi_{in}(x_n) U(\infty)\}] | 0 \rangle}{\langle 0 | U(\infty) | 0 \rangle}, \quad (\text{C.17})$$

where

$$U(\infty) = U(\tau, -\tau), \quad (\text{C.18})$$

when τ tends to infinity. The perturbation expansion is obtained by expanding $U(\infty)$ and reducing the products of field operators through Wick's theorem.

C.2.2 Interacting Scalar Fields in Presence of a Thermal Bath.

When there is a thermal bath present, then

$$\lim_{\tau \rightarrow \infty} U(-\tau)|0, \beta\rangle = |0, \beta\rangle,$$

(where $|0, \beta\rangle$ is the thermal vacuum) but

$$\lim_{\tau \rightarrow \infty} U(\tau)|0, \beta\rangle \neq e^{i\delta}|0, \beta\rangle,$$

as there are real particles inside the thermal bath (thermal vacuum), they can interact and the state may change physically. As a result $\langle 0, \beta | U^{-1}(\tau)$ cannot be just replaced by $e^{-i\delta}\langle 0, \beta |$ as is done in Eq. (C.17). Here

$$G(x_1, \cdot, \cdot, \cdot, x_n) = \langle U^{-1}(\infty) [T\{\phi_{in}(x_1) \cdots \phi_{in}(x_n) U(\infty)\}] \rangle. \quad (\text{C.19})$$

The expectation value of some operator O is defined as

$$\langle O \rangle = \frac{\text{Tr } Z O}{\text{Tr } Z}, \quad (\text{C.20})$$

where Tr indicates a trace has to be taken (with respect to any basis) and Z is the partition function given by

$$Z = \exp \left[-\beta H + \sum_A \alpha_A Q_A \right], \quad (\text{C.21})$$

where Q_A are the conserved charges that commute with the Lagrangian and α_A are the chemical potentials that parameterize the composition of the medium. Here we are working in presence of a thermal bath, and it can have a velocity. For the next part of the discussions we will work in the rest frame of the heat bath.

The perturbation expansion is obtained by expanding the formal formula for $U(\infty)$, but in this case we have to take into account the factor $U^{-1}(\infty)$, which is given by

$$U^{-1}(\infty) = \overline{T} \exp \left[-i \int d^4x \mathcal{L}_{int}[\phi_{in}(x)] \right], \quad (\text{C.22})$$

and expand it also. \overline{T} indicates anti-time ordered product.

C.2.3 Calculation of the Thermal Green's Function.

Writing $\mathcal{L}_{int}[\phi_{in}(x)]$ as $\mathcal{L}_{int}(x)$ for brevity, the whole expression of the Green's function can be written as,

$$\begin{aligned} G(x_1, \cdot, \cdot, \cdot, x_n) &= \sum_{s=0}^{\infty} \frac{(-i)^s}{s!} \sum_{p=0}^{\infty} \frac{(i)^p}{p!} \langle \overline{T} \left[\int d^4z_1 \int d^4z_2 \cdots \int d^4z_s \mathcal{L}_{int}(z_1) \mathcal{L}_{int}(z_2) \cdots \mathcal{L}_{int}(z_s) \right] \right. \\ &\quad \times \left. T \left[\phi_{in}(x_1) \cdots \phi_{in}(x_n) \int d^4y_1 \cdots d^4y_p \mathcal{L}_{int}(y_1) \cdots \mathcal{L}_{int}(y_p) \right] \right\rangle. \end{aligned} \quad (\text{C.23})$$

Concentrating on ϕ^4 theory where the interaction term is given by

$$\mathcal{L}_{int}(x) = -\frac{\lambda}{4!} \phi_{in}^4(x),$$

we can write the Green's function as

$$\begin{aligned} G(x_1, \cdot, \cdot, \cdot, x_n) &= \sum_{s=0}^{\infty} \frac{(i\lambda)^s}{s!} \sum_{p=0}^{\infty} \frac{(-i\lambda)^p}{p!} \left\langle \overline{T} \left[\int d^4z_1 \int d^4z_2 \cdots \int d^4z_s \frac{\phi_{in}^4(z_1)}{4!} \cdots \frac{\phi_{in}^4(z_s)}{4!} \right] \right. \\ &\quad \times \left. T \left[\phi_{in}(x_1) \cdots \phi_{in}(x_n) \int d^4y_1 \cdots d^4y_p \frac{\phi_{in}^4(y_1)}{4!} \cdots \frac{\phi_{in}^4(y_p)}{4!} \right] \right\rangle \\ &= \sum_{s,p=0}^{\infty} \frac{(i\lambda)^s (-i\lambda)^p}{s! p!} \int d^4z_1 \cdots \int d^4z_s \int d^4y_1 \cdots \int d^4y_p \\ &\quad \times \left\langle \left[\overline{T} \left\{ \frac{\phi_{in}^4(z_1)}{4!} \cdots \frac{\phi_{in}^4(z_s)}{4!} \right\} T \left\{ \phi_{in}(x_1) \cdots \phi_{in}(x_n) \frac{\phi_{in}^4(y_1)}{4!} \cdots \frac{\phi_{in}^4(y_p)}{4!} \right\} \right] \right\rangle. \end{aligned} \quad (\text{C.24})$$

This expansion shows that the general expectation value term will be of the form

$$\langle \overline{T}(\overline{A} \overline{B} \cdots \overline{F}) T(A B \cdots F) \rangle \quad (\text{C.25})$$

where the bars over A, B, \dots indicate they are operators under the \bar{T} sign, or they are anti-time ordered operators. The objects A, B, \dots are the fields that appear in the last line of the Eq. (C.24). The vertices that come under T are called the type-1 vertices, and those that come under \bar{T} are called the type-2 vertices. A propagator which joins two vertices can now be of four kinds,

- 1-1 type, which joins two vertices under the T sign.
- 2-2 type, which joins two vertices under the \bar{T} sign.
- 1-2 type, which joins a type-1 vertex to a type-2 vertex.
- 2-1 type, which joins a type-2 vertex to a type-1 vertex.

Another important thing to be noticed is, any external vertex where real particles come in or go out must be of type-1 variety, as those fields $\phi(x)$ s are under the T operation in Eq. (C.24). If now we calculate the two point function where both the fields are of type-1 by simplifying the Eq. (C.25) properly then we will arrive at Eq. (C.8). But there can be other options as well, where the fields are both of type-2, or are mixed. Not going into further details here the other propagators are supplied for the scalar theory. $\Delta_{F11}(k)$ is given by Eq. (C.8), the others are

$$\Delta_{F22}(p) = \frac{-1}{p^2 - m^2 + i\varepsilon} - 2\pi i\delta(p^2 - m^2)\eta(p \cdot u), \quad (\text{C.26})$$

$$\Delta_{F21}(p) = -2\pi i\delta(p^2 - m^2)[\eta(p \cdot u) + \theta(-p \cdot u)], \quad (\text{C.27})$$

$$\Delta_{F12}(p) = -2\pi i\delta(p^2 - m^2)[\eta(p \cdot u) + \theta(p \cdot u)], \quad (\text{C.28})$$

where

$$\eta(p \cdot u) = \theta(p \cdot u)n_B(x) + \theta(-p \cdot u)n_B(-x), \quad (\text{C.29})$$

with

$$n_B(x) = \frac{1}{e^x - 1}, \quad (\text{C.30})$$

and x is defined as,

$$x = \beta p \cdot u. \quad (\text{C.31})$$

For fermions the procedure is same, except here we have anti-commutation relations in place of commutation relations, and the expressions for the four propagators are,

$$S_{F11}(p) = (\not{p} + m) \left[\frac{1}{p^2 - m^2 + i\varepsilon} + 2\pi i \delta(p^2 - m^2) \eta_F(p \cdot u) \right], \quad (\text{C.32})$$

$$S_{F22}(p) = (\not{p} + m) \left[\frac{-1}{p^2 - m^2 + i\varepsilon} + 2\pi i \delta(p^2 - m^2) \eta_F(p \cdot u) \right], \quad (\text{C.33})$$

$$S_{F12}(p) = (\not{p} + m) 2\pi i \delta(p^2 - m^2) [\eta_F(p \cdot u) - \theta(-p \cdot u)], \quad (\text{C.34})$$

$$S_{F21}(p) = (\not{p} + m) 2\pi i \delta(p^2 - m^2) [\eta_F(p \cdot u) - \theta(p \cdot u)]. \quad (\text{C.35})$$

In the case of fermions $\eta_F(p \cdot u)$ is given by,

$$\eta_F(p \cdot u) = \Theta(p \cdot u) f_F(p, \mu, \beta) + \Theta(-p \cdot u) f_F(-p, -\mu, \beta), \quad (\text{C.36})$$

where,

$$f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p \cdot u - \mu)} + 1}, \quad (\text{C.37})$$

is the Fermi-Dirac distribution function.

With this brief exposition of statistical field theory in the real time formalism we understand why instead of one propagator we have four of them. But usually most of the time we only work with the 1-1 component as is done in this thesis. But formal calculations in statistical field theory requires all four propagators as specified above.

APPENDIX D

Manipulations under the integral sign

All the results appearing in this appendix have been derived previously in [30]. They are explicitly written down here because the following techniques are very useful in manipulating the integrand inside the integral sign in an equation like Eq. (4.59). If we concentrate on the rest frame of the medium, then $p \cdot u = p_0$. Thus, the distribution function does not depend on the spatial components of p . From the form of Eq. (4.60) and Eq. (4.61) we find that, the integral over the transverse components of p has the following generic structure:

$$\int d^2 p_\perp e^{\Phi(p,s)} e^{\Phi(p',s')} \times (p^{\beta_\perp} \text{ or } p'^{\beta_\perp}). \quad (\text{D.1})$$

Notice now that

$$\frac{\partial}{\partial p_{\beta_\perp}} \left[e^{\Phi(p,s)} e^{\Phi(p',s')} \right] = \frac{2i}{e\mathcal{B}} \left(\tan(e\mathcal{B}s) p^{\beta_\perp} + \tan(e\mathcal{B}s') p'^{\beta_\perp} \right) e^{\Phi(p,s)} e^{\Phi(p',s')}. \quad (\text{D.2})$$

However, this expression, being a total derivative, should integrate to zero. Thus we obtain

$$\int d^2 p_\perp \tan(e\mathcal{B}s) p^{\beta_\perp} e^{\Phi(p,s)} e^{\Phi(p',s')} = - \int d^2 p_\perp \tan(e\mathcal{B}s') p'^{\beta_\perp} e^{\Phi(p,s)} e^{\Phi(p',s')}. \quad (\text{D.3})$$

This above equation is symbolically written as,

$$\tan(e\mathcal{B}s) p^{\beta_\perp} \stackrel{\circ}{=} - \tan(e\mathcal{B}s') p'^{\beta_\perp}, \quad (\text{D.4})$$

where the sign ‘ $\stackrel{\circ}{=}$ ’ means that the expressions on both sides of it, though not necessarily equal algebraically, yield the same integral. This gives

$$p^{\beta_\perp} \stackrel{\circ}{=} - \frac{\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k^{\beta_\perp}, \quad (\text{D.5})$$

$$p'^{\beta_\perp} \stackrel{\circ}{=} \frac{\tan(e\mathcal{B}s)}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k^{\beta_\perp}. \quad (\text{D.6})$$

Similarly we can derive some other relations which can be used under the momentum integral signs. To write them in a useful form, we turn to Eq. (D.2) and take another derivative with respect to $p^{\alpha\perp}$. From the fact that this derivative should also vanish on p integration, we find

$$p_\perp^\alpha p_\perp^\beta \stackrel{\circ}{=} \frac{1}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \left[\frac{ie\mathcal{B}}{2} g_\perp^{\alpha\beta} + \frac{\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_\perp^\alpha k_\perp^\beta \right]. \quad (\text{D.7})$$

In particular, then,

$$p_\perp^2 \stackrel{\circ}{=} \frac{1}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \left[-ie\mathcal{B} + \frac{\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_\perp^2 \right]. \quad (\text{D.8})$$

It then simply follows that

$$p'^2_\perp \stackrel{\circ}{=} \frac{1}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \left[-ie\mathcal{B} + \frac{\tan^2(e\mathcal{B}s)}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_\perp^2 \right]. \quad (\text{D.9})$$

And finally using the definition of the exponential factor in Eq. (3.4) we can write

$$m^2 \stackrel{\circ}{=} \left(i \frac{d}{ds} + (p_\parallel^2 - \sec^2(e\mathcal{B}s)p_\perp^2) \right). \quad (\text{D.10})$$

Here the ‘ $\stackrel{\circ}{=}$ ’ means that both sides of the above equation, when multiplied by $e^{\Phi(p,s)}e^{\Phi(p',s')}$, are equivalent inside the momentum integrals.

This set of relations help us considerably to handle the terms present inside the integrals appearing in Eq. (4.61) and Eq. (4.60).

APPENDIX E

A list of relevant integrals

These kinds of integrals often crop up when we are calculating with the Schwinger propagator.

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{iax^2} = \frac{e^{i\pi/4}}{2\sqrt{\pi a}}, \quad (\text{E.1})$$

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-iax^2} = \frac{e^{-i\pi/4}}{2\sqrt{\pi a}}. \quad (\text{E.2})$$

Here a is a real number. Obviously the second integral can be obtained from complex conjugating the first one.

Next we consider integrals containing vector indices.

$$\int \frac{d^4 p}{(2\pi)^4} e^{-ia(p-B)^2} = \frac{i}{16\pi^2 a^2}, \quad (\text{E.3})$$

$$\int \frac{d^4 p}{(2\pi)^4} p_\mu e^{-ia(p-B)^2} = \frac{i}{16\pi^2 a^2} B_\mu, \quad (\text{E.4})$$

$$(\text{E.5})$$

Here, as before a is a real number, p and B are 4-vectors. For each one of the above integrals there exists one corresponding complex conjugate integral. The set of complex conjugated integrals is not written here.

Finally comes those integrals which involve the parallel and perpendicular components of the vectors.

$$\int \frac{d^2 p_\parallel}{(2\pi)^2} e^{-ia(p-B)_\parallel^2} = \frac{1}{4\pi a}, \quad (\text{E.6})$$

$$\int \frac{d^2 p_\perp}{(2\pi)^2} e^{-ia(p-B)_\perp^2} = -\frac{i}{4\pi a}, \quad (\text{E.7})$$

$$\int \frac{d^2 p_\parallel}{(2\pi)^2} p_{\mu_\parallel} e^{-ia(p-B)_\parallel^2} = \frac{1}{4\pi a} B_{\mu_\parallel}, \quad (\text{E.8})$$

$$\int \frac{d^2 p_\perp}{(2\pi)^2} p_{\mu_\perp} e^{-ia(p-B)_\perp^2} = -\frac{i}{4\pi a} B_{\mu_\perp}, \quad (\text{E.9})$$

$$(\text{E.10})$$

Here also we have one corresponding complex conjugated integral for each one of the above integrals which are not written.

APPENDIX F

Some calculations in a magnetized medium

F.1 Derivation of $R_{\mu\nu}^{(e)}$ and $R_{\mu\nu}^{(o)}$

In this appendix those calculations are done which were side stepped in chapter 4. First it is shown how we get Eq. (4.65) from Eq. (4.60) and how Eq. (4.66) is derived. From Eq. (4.60) it is clear that if we apply Eq. (D.7) and then Eq. (D.5) to the last term in the right hand side then it drops out and we get Eq. (4.65).

An expression of $R_{\mu\nu}^{(o)}$ is given in Eq. (4.61), leaving it aside we derive Eq. (4.66) in a different way. Now from Eq. (4.59) it is known that in the rest frame of the medium,

$$R_{\mu\nu}(p, p', s, s') = \text{Tr}[\gamma_\mu \gamma_5 G(p, s) \gamma_\nu G(p', s')] \eta_F(p_0) + \text{Tr}[\gamma_\mu \gamma_5 G(-p', s') \gamma_\nu G(-p, s)] \eta_F(-p_0). \quad (\text{F.1})$$

$R_{\mu\nu}(p, p', s, s')$ contains the sum of two traces multiplied by the functions $\eta_F(\pm p_0)$. Except the η functions, the two traces are equal, so for the part which is odd in orders of the external field, Eq. (F.1) is equivalent to,

$$R_{\mu\nu}^{(o)} = \eta_+(p_0) \text{Tr}[T_{\mu\nu}^{(o)}] \quad (\text{F.2})$$

where for brevity the arguments of $R_{\mu\nu}^{(o)}$ and $T_{\mu\nu}^{(o)}$ are suppressed. Here $\eta_+(p_0) = \eta(p_0) + \eta(-p_0)$ and $T_{\mu\nu}^{(o)}$ is given by,

$$\begin{aligned} T_{\mu\nu}^{(o)} &= i \tan(eBs) \left[m^2 \gamma_\mu \gamma_5 \sigma_z \gamma_\nu + \gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel \gamma_\nu \not{p}'_\parallel + \gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel \gamma_\nu \not{p}'_\perp \sec^2(eBs') \right] \\ &\quad + i \tan(eBs') \left[m^2 \gamma_\mu \gamma_5 \gamma_\nu \sigma_z + \gamma_\mu \gamma_5 \not{p}_\parallel \gamma_\nu \sigma_z \not{p}'_\parallel + \gamma_\mu \gamma_5 \not{p}_\perp \gamma_\nu \sigma_z \not{p}'_\parallel \sec^2(eBs) \right]. \end{aligned} \quad (\text{F.3})$$

Writing out the terms in $T_{\mu\nu}^{(o)}$, we get

$$\begin{aligned}\text{Tr}[\gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel \gamma_\nu \not{p}'_\parallel] &= \text{Tr} \left[2p'_{\nu\parallel} \gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel - p_\parallel^2 \gamma_\nu \gamma_\mu \gamma_5 \sigma_z - (p \cdot k)_\parallel \gamma_\nu \gamma_\mu \gamma_5 \sigma_z \right. \\ &\quad \left. + (p^0 k^3 - p^3 k^0) \gamma_\nu \gamma_\mu \right],\end{aligned}\tag{F.4}$$

$$\begin{aligned}\text{Tr}[\gamma_\mu \gamma_5 \not{p}_\parallel \gamma_\nu \sigma_z \not{p}'_\parallel] &= \text{Tr} \left[2p_{\nu\parallel} \gamma_\mu \gamma_5 \sigma_z \not{p}'_\parallel + p_\parallel^2 \gamma_\mu \gamma_\nu \gamma_5 \sigma_z + (p \cdot k)_\parallel \gamma_\mu \gamma_\nu \gamma_5 \sigma_z \right. \\ &\quad \left. - (p^0 k^3 - p^3 k^0) \gamma_\mu \gamma_\nu \right].\end{aligned}\tag{F.5}$$

Using Eq. (F.4) and Eq. (F.5) we can write Eq. (F.3) as,

$$\begin{aligned}\text{Tr}[T_{\mu\nu}^{(o)}] &= i \tan(e\mathcal{B}s) \text{Tr} \left[\{m^2 - p_\parallel^2 - (p \cdot k)_\parallel\} \gamma_\nu \gamma_\mu \gamma_5 \sigma_z + (p^0 k^3 - p^3 k^0) \gamma_\nu \gamma_\mu \right. \\ &\quad + 2p'_{\nu\parallel} \gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel + \gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel \gamma_\nu \not{p}'_\perp \sec^2(e\mathcal{B}s') \Big] \\ &\quad + i \tan(e\mathcal{B}s') \text{Tr} \left[-\{m^2 - p_\parallel^2 - (p \cdot k)_\parallel\} \gamma_\mu \gamma_\nu \gamma_5 \sigma_z - (p^0 k^3 - p^3 k^0) \gamma_\mu \gamma_\nu \right. \\ &\quad \left. + 2p_{\nu\parallel} \gamma_\mu \gamma_5 \sigma_z \not{p}'_\parallel + \gamma_\mu \gamma_5 \not{p}_\perp \gamma_\nu \sigma_z \not{p}'_\parallel \sec^2(e\mathcal{B}s) \right].\end{aligned}\tag{F.6}$$

As we are interested in taking the trace of $T_{\mu\nu}^{(o)}$, so now we write down the traces of the components of $T_{\mu\nu}^{(o)}$. The easier ones comes first, and they are

$$\text{Tr}[\gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel] = 4\varepsilon_{\mu 12\alpha\parallel} p^{\alpha\parallel},\tag{F.7}$$

$$\text{Tr}[\gamma_\mu \gamma_5 \sigma_z \not{p}'_\parallel] = 4\varepsilon_{\mu 12\alpha\parallel} p'^{\alpha\parallel},\tag{F.8}$$

$$\begin{aligned}\text{Tr}[\gamma_\mu \gamma_\nu \gamma_5 \sigma_z] &= -\text{Tr}[\gamma_\nu \gamma_\mu \gamma_5 \sigma_z], \\ &= -4\varepsilon_{\mu\nu 12}.\end{aligned}\tag{F.9}$$

Next comes the trace of a part of $T_{\mu\nu}^{(o)}$, and it is

$$\begin{aligned}&\text{Tr} \left[\gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel \gamma_\nu \not{p}'_\perp \tan(e\mathcal{B}s) \sec^2(e\mathcal{B}s') + \gamma_\mu \gamma_5 \not{p}_\perp \gamma_\nu \sigma_z \not{p}'_\parallel \tan(e\mathcal{B}s') \sec^2(e\mathcal{B}s) \right] \\ &= 4g_{\mu\alpha\parallel} \left[p^{\tilde{\alpha}\parallel} p'_{\nu\perp} \sec^2(e\mathcal{B}s') \tan(e\mathcal{B}s) + p'^{\tilde{\alpha}\parallel} p_{\nu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right] \\ &\quad + 4g_{\nu\alpha\parallel} \left[p^{\tilde{\alpha}\parallel} p'_{\mu\perp} \sec^2(e\mathcal{B}s') \tan(e\mathcal{B}s) + p'^{\tilde{\alpha}\parallel} p_{\mu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right] \\ &\quad + 4g_{\nu\alpha\parallel} k^{\tilde{\alpha}\parallel} p_{\mu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s').\end{aligned}\tag{F.10}$$

Now using Eq. (D.5) and Eq. (D.6) from appendix D, the above trace can be rewritten as,

$$\begin{aligned}&\text{Tr} \left[\gamma_\mu \gamma_5 \sigma_z \not{p}_\parallel \gamma_\nu \not{p}'_\perp \tan(e\mathcal{B}s) \sec^2(e\mathcal{B}s') + \gamma_\mu \gamma_5 \not{p}_\perp \gamma_\nu \sigma_z \not{p}'_\parallel \tan(e\mathcal{B}s') \sec^2(e\mathcal{B}s) \right] \\ &= 4(g_{\mu\alpha\parallel} p^{\tilde{\alpha}\parallel} k_{\nu\perp} + g_{\nu\alpha\parallel} p^{\tilde{\alpha}\parallel} k_{\mu\perp}) \{ \tan(e\mathcal{B}s) - \tan(e\mathcal{B}s') \} \\ &\quad - 4g_{\mu\alpha\parallel} k_{\nu\perp} k^{\tilde{\alpha}\parallel} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} + 4g_{\nu\alpha\parallel} k^{\tilde{\alpha}\parallel} p_{\mu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s').\end{aligned}\tag{F.11}$$

Combining the previous steps, the trace of $T_{\mu\nu}^{(o)}$ can be written as,

$$\begin{aligned}
\text{Tr}[T_{\mu\nu}^{(o)}] &= 4i \left[\varepsilon_{\mu\nu 12} \{m^2 - p_{\parallel}^2 - (p \cdot k)_{\parallel}\} \{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')\} \right. \\
&\quad + 2\varepsilon_{\mu 12\alpha_{\parallel}} \{p'_{\nu_{\parallel}} p^{\alpha_{\parallel}} \tan(e\mathcal{B}s) + p_{\nu_{\parallel}} p'^{\alpha_{\parallel}} \tan(e\mathcal{B}s')\} \\
&\quad + g_{\mu\nu} (p^0 k^3 - p^3 k^0) \{\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')\} \\
&\quad + \tilde{p}^{\alpha_{\parallel}} (g_{\mu\alpha_{\parallel}} k_{\nu_{\perp}} + g_{\nu\alpha_{\parallel}} k_{\mu_{\perp}}) \{\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')\} \\
&\quad \left. + g_{\mu\alpha_{\parallel}} k^{\tilde{\alpha}_{\parallel}} k_{\nu_{\perp}} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} + g_{\nu\alpha_{\parallel}} k^{\tilde{\alpha}_{\parallel}} p_{\mu_{\perp}} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right]. \tag{F.12}
\end{aligned}$$

Now we modify the term $(m^2 - p_{\parallel}^2) \{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')\}$ in the first line of the right hand side of the above equation. As the trace of $T_{\mu\nu}^{(o)}$ is related to $R_{\mu\nu}^{(o)}$ by Eq. (F.2), and as $R_{\mu\nu}^{(o)}$ is inside the momentum integrals in Eq. (4.59) in chapter 4 so we can use the operations under the integral signs. The operations were explicitly written down in appendix D. Using Eq. (D.10) we can write,

$$\begin{aligned}
(m^2 - p_{\parallel}^2) \{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')\} e^{[\Phi(p,s) + \Phi(p',s')]} \\
\stackrel{\circ}{=} \{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')\} \left[i \frac{d}{ds} - \sec^2(e\mathcal{B}s) p_{\perp}^2 \right] e^{[\Phi(p,s) + \Phi(p',s')]},
\end{aligned}$$

where $e^{[\Phi(p,s) + \Phi(p',s')]}$ is the overall phase factor appearing inside the momentum integrals in Eq. (4.59). Using Eq. (D.8) the above equation becomes,

$$\begin{aligned}
(m^2 - p_{\parallel}^2) \{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')\} e^{[\Phi(p,s) + \Phi(p',s')]} \\
\stackrel{\circ}{=} \left[\{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')\} \left(i \frac{d}{ds} \right) \right. \\
\left. - \sec^2(e\mathcal{B}s) \left\{ -ieB + \frac{\tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_{\perp}^2 \right\} \right] e^{[\Phi(p,s) + \Phi(p',s')]}.
\tag{F.13}
\end{aligned}$$

The above equation is valid under the momentum integrals and the parametric integrals in Eq. (4.59), so we can do some of the parametric integrals right now to modify Eq. (F.13). First of all,

$$\begin{aligned}
\int_{-\infty}^{\infty} \tan(e\mathcal{B}s) \frac{d}{ds} e^{[\Phi(p,s) + \Phi(p',s')]} ds &= \tan(e\mathcal{B}s') e^{\Phi(p',s')} \left[e^{\Phi(p,s)} \right]_{-\infty}^{\infty}, \\
&= 0,
\tag{F.14}
\end{aligned}$$

then,

$$i \int_{-\infty}^{\infty} \tan(e\mathcal{B}s) \frac{d}{ds} e^{[\Phi(p,s) + \Phi(p',s')]} ds = -ie\mathcal{B} \int_{-\infty}^{\infty} \sec^2(e\mathcal{B}s) e^{[\Phi(p,s) + \Phi(p',s')]} ds,
\tag{F.15}$$

and so combining the above steps we get from Eq. (F.13),

$$(m^2 - p_{\parallel}^2) \{ \tan(e\mathcal{B}s) + \tan(e\mathcal{B}s') \} e^{[\Phi(p,s) + \Phi(p',s')]} \stackrel{\circ}{=} - \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_{\perp}^2 \\ \times e^{[\Phi(p,s) + \Phi(p',s')]} . \quad (\text{F.16})$$

Now from Eq. (F.12) and Eq. (F.16), we can write

$$\begin{aligned} \text{Tr}[T_{\mu\nu}^{(o)}] &\stackrel{\circ}{=} 4i \left[-\varepsilon_{\mu\nu 12} \left\{ \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_{\perp}^2 + (k \cdot p)_{\parallel} \{ \tan(e\mathcal{B}s) + \tan(e\mathcal{B}s') \} \right\} \right. \\ &+ 2\varepsilon_{\mu 12\alpha_{\parallel}} (p'_{\nu_{\parallel}} p^{\alpha_{\parallel}} \tan(e\mathcal{B}s) + p_{\nu_{\parallel}} p'^{\alpha_{\parallel}} \tan(e\mathcal{B}s')) \\ &+ g_{\mu\alpha_{\parallel}} k_{\nu\perp} \left\{ p^{\tilde{\alpha}_{\parallel}} \{ \tan(e\mathcal{B}s) - \tan(e\mathcal{B}s') \} - k^{\tilde{\alpha}_{\parallel}} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right\} \\ &+ \{ g_{\mu\nu} (p \cdot \tilde{k})_{\parallel} + g_{\nu\alpha_{\parallel}} p^{\tilde{\alpha}_{\parallel}} k_{\mu\perp} \} \{ \tan(e\mathcal{B}s) - \tan(e\mathcal{B}s') \} \\ &\left. + g_{\nu\alpha_{\parallel}} k^{\tilde{\alpha}_{\parallel}} p_{\mu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right]. \end{aligned} \quad (\text{F.17})$$

From the above equation and Eq. (F.2) we immediately see that $R_{\mu\nu}^{(o)}$ matches with Eq. (4.66) in chapter 4.

F.2 Derivation of the expression of Π_{30}^5

First of all it is shown why the other components of $\Pi_{\mu 0}^5$ except Π_{30}^5 turns out to be zero in the limit when the external momentum goes to zero. First the contributions from $\Pi_{\mu 0}^{5(o)}$ are considered. From Eq. (4.59) and Eq. (4.66) we can write

$$\begin{aligned} \Pi_{00}^{5(o)} &= 4e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ &\times \int_0^{\infty} ds' e^{\Phi(p',s')} \eta_+(p_0) \left[2\varepsilon_{0123} \{ p'_0 p^3 \tan(e\mathcal{B}s) + p_0 p'^3 \tan(e\mathcal{B}s') \} \right. \\ &\left. + g_{00} (p \cdot \tilde{k})_{\parallel} \{ \tan(e\mathcal{B}s) - \tan(e\mathcal{B}s') \} \right]. \end{aligned} \quad (\text{F.18})$$

$e^{[\Phi(p,s) + \Phi(p',s')]} \eta_+(p_0)$ is an even function of p_0 when $k_0 = 0$. $\eta_+(p_0)$ inside the integrand is also an even function of p_0 . From these observations it can be said that when $k_0 = 0$ the integrand on the right hand side of the above equation becomes an odd function of p_0 and as a result the integral vanishes.

Then comes $\Pi_{10}^{5(o)}$ and it is,

$$\begin{aligned} \Pi_{10}^{5(o)} &= -4e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ &\times \int_0^{\infty} ds' e^{\Phi(p',s')} \eta_+(p_0) \left[p^3 k^1 \{ \tan(e\mathcal{B}s) - \tan(e\mathcal{B}s') \} + k^3 p^1 \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right]. \end{aligned} \quad (\text{F.19})$$

$\Pi_{10}^{5(o)}$ can exist only when the parametric integrals can yield terms which cancels terms like k^1 or k^3 . In the present case as this does not happen so due to the presence of the terms like k^3 and k^1 , the integral vanishes in the limit $\mathbf{k} \rightarrow 0$. Similarly $\Pi_{20}^{5(o)}$ will also be zero.

Now comes the contributions from $\Pi_{\mu 0}^{5(e)}$. From the expression of $R_{\mu\nu}^{(e)}$ in Eq. (4.65) we can immediately say that $\Pi_{\mu\parallel 0}^{5(e)}$ must be zero. If on the other hand $\mu = 1$, then

$$\begin{aligned}\Pi_{10}^{5(e)} &= 4e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ &\times \int_0^{\infty} ds' e^{\Phi(p',s')} \eta_+(p_0) \varepsilon_{10\alpha\beta} \{p^{\alpha\parallel} p'^{\beta\perp} \sec^2(e\mathcal{B}s') + p^{\alpha\perp} p'^{\beta\parallel} \sec^2(e\mathcal{B}s)\},\end{aligned}\quad (\text{F.20})$$

and the above equation shows that each term in the right hand side inside the momentum integrals contain one perpendicular component of loop momentum, and so it must integrate out to perpendicular components of the external momentum \mathbf{k} . $\Pi_{10}^{5(e)}$ can exist when the parametric integrals can produce terms which will cancel the perpendicular components of \mathbf{k} . As in the present case it does not happen as a result the integral vanishes when $\mathbf{k} \rightarrow 0$. Similar result is obtained for $\Pi_{20}^{5(e)}$. Ultimately the only component of $\Pi_{\mu 0}^{5(e)}$ which is non vanishing in the specified external momentum limit is $\Pi_{30}^{5(o)}$, which is simply written as Π_{30}^5 .

Now the relevant steps are given by which we get Eq. (4.75) from Eq. (4.74). To begin with we explicitly write down the sum of the functions $\Phi(p,s)$ and $\Phi(p',s')$ in terms of the momentums p and p' .

$$\begin{aligned}\Phi(p,s) + \Phi(p',s') &= isp_{\parallel}^2 + is'p'^{\prime 2}_{\parallel} - i(s+s')m^2 \\ &- \frac{i}{e\mathcal{B}} \{ \tan(e\mathcal{B}s)p_{\perp}^2 + \tan(e\mathcal{B}s')p'^{\prime 2}_{\perp} \} - \varepsilon|s| - \varepsilon|s'|.\end{aligned}\quad (\text{F.21})$$

Now we concentrate on the integration of the perpendicular components of the loop momentum in Eq. (4.74), and try to see what happens when $|k_{\perp}| \rightarrow 0$. It must be noted that we have thrown away some of the terms in deriving the expression of $\Pi_{\mu 0}^{5(e)}$ by assuming that $k_0 = 0$ and $\mathbf{k} \rightarrow 0$. The terms still existing in Eq. (4.74) do not vanish in the above specified momentum limit. First we integrate out the perpendicular components of momenta in the integral appearing in Eq. (4.74). As the integrand in the right hand side of Eq. (4.74) does not contain any term involving the perpendicular momentum components, except the phase like terms, so here the integration is straight forward. From Eq. (F.21) if we only take the terms containing the squares of the perpendicular components of p

and p' , and try to integrate them out, using the results given in appendix E, we will get

$$\begin{aligned} & \int \frac{d^2 p_\perp}{(2\pi)^2} \exp \left\{ -\frac{i}{e\mathcal{B}} [\tan(e\mathcal{B}s) p_\perp^2 + \tan(e\mathcal{B}s') p_\perp'^2] \right\} \\ &= \exp \left\{ -\frac{ik_\perp^2}{e\mathcal{B}} \left[\frac{\tan(e\mathcal{B}s) \tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right] \right\} \\ & \times \int \frac{d^2 p_\perp}{(2\pi)^2} \exp \left\{ -\frac{i}{e\mathcal{B}} [\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')] \left[p_\perp + k_\perp \frac{\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right]^2 \right\}, \end{aligned} \quad (\text{F.22})$$

and as $k_\perp^2 \rightarrow 0$ the above equation simplifies to,

$$\int \frac{d^2 p_\perp}{(2\pi)^2} \exp \left\{ -\frac{i}{e\mathcal{B}} [\tan(e\mathcal{B}s) p_\perp^2 + \tan(e\mathcal{B}s') p_\perp'^2] \right\} = -\frac{1}{4\pi} \frac{ie\mathcal{B}}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} . \quad (\text{F.23})$$

Using the above equation we get Eq. (4.75) from Eq. (4.74).

F.3 Effective charge calculation

This section gives the details of the calculations by which the expression of neutrino effective charge was calculated in chapter 4.

The s integral in Eq. (4.75) gives

$$\int_{-\infty}^{\infty} ds e^{is(p_\parallel^2 - m^2) - \varepsilon|s|} = 2\pi \delta(p_\parallel^2 - m^2) \quad (\text{F.24})$$

and the s' integral gives

$$\int_0^{\infty} ds' e^{is'(p_\parallel'^2 - m^2) - \varepsilon|s'|} = \frac{i}{(p_\parallel'^2 - m^2) + i\varepsilon}. \quad (\text{F.25})$$

Using the above results in Eq.(4.75) and using the delta function constraint, we arrive at,

$$\Pi_{30}^5 = \lim_{k_0=0, \mathbf{k} \rightarrow 0} -2(e^3 \mathcal{B}) \int \frac{d^2 p_\parallel}{(2\pi)^2} \delta(p_\parallel^2 - m^2) \eta_+(p_0) \left[\frac{2p_0^2}{(k_\parallel^2 + 2(p.k)_\parallel)} - \frac{1}{2} \right]. \quad (\text{F.26})$$

In deriving Eq. (F.26), pieces proportional to k_\parallel^2 in the numerator were neglected. Now if one makes the substitution, $p'_\parallel \rightarrow (p_\parallel + k_\parallel/2)$ and sets $k_0 = 0$ one arrives at,

$$\Pi_{30}^5 = \lim_{k_0=0, \mathbf{k} \rightarrow 0} 2(e^3 \mathcal{B}) \int \frac{dp_3}{(2\pi)^2} \left(n_+(E'_p) + n_-(E'_p) \right) \left[\frac{E'_p}{p_3 k_3} + \frac{1}{2E'_p} \right]. \quad (\text{F.27})$$

Here $n_\pm(E'_p)$ are the functions $f_F(E'_p, \mu, \beta)$, and $f_F(E'_p, -\mu, \beta)$, as given in Eq.(3.19), which are nothing but the Fermi-Dirac distribution functions of the particles and the

antiparticles in the medium with a modified energy E'_p . The new term E'_p is defined as follows,

$$E'^2_p = [(p_3 - k_3/2)]^2 + m^2,$$

and it can be expanded for small external momenta in the following way

$$E'^2_p \simeq p_3^2 + m^2 - p_3 k_3 = E_p^2 - p_3 k_3$$

where $E_p^2 = p_3^2 + m^2$. Noting, that

$$E'_p = E_p - \frac{p_3 k_3}{2 E_p} + O(k_3^2), \quad (\text{F.28})$$

one can use this expansion in Eq. (F.27), to arrive at:

$$\Pi_{30}^5 = \lim_{k_0=0, \mathbf{k} \rightarrow 0} 2(e^3 \mathcal{B}) \int \frac{dp_3}{(2\pi)^2} \left(n_+(E'_p) + n_-(E'_p) \right) \left[\frac{E_p}{p_3 k_3} \right]. \quad (\text{F.29})$$

In the classical gas limit the expression for $\eta_+(E'_p) = n_+(E'_p) + n_-(E'_p)$ when expanded in powers of the external momentum k_3 is given by

$$\eta_+(E'_p) = (1 + \frac{1}{2} \frac{\beta p_3 k_3}{E_p}) \eta_+(E_p), \quad (\text{F.30})$$

up to first order terms in the external momentum k_3 .

From Eq. (F.30) and Eq. (F.29) we get,

$$\Pi_{30}^5 = \lim_{k_0=0, \mathbf{k} \rightarrow 0} 2(e^3 \mathcal{B}) \int \frac{dp_3}{(2\pi)^2} \left(\frac{E_p}{p_3 k_3} + \frac{\beta}{2} \right) \eta_+(E_p). \quad (\text{F.31})$$

The first term in the integral on the right hand side of the above equation contains a factor of p_3 in the denominator and so is an odd function of p_3 . As a result the integral vanishes by symmetry. But the second integral survives and Π_{30}^5 is given by,

$$\Pi_{30}^5 = \frac{e^3 \mathcal{B}}{2\pi} \beta \int \frac{dp}{2\pi} \eta_+(E_p). \quad (\text{F.32})$$

In the classical limit:

$$\Pi_{30}^5 = \frac{e^3 \mathcal{B}}{2\pi^2} \beta \cosh(\beta\mu) \int dp e^{-\beta E_p}. \quad (\text{F.33})$$

Using the standard result [92],

$$\int_0^\infty e^{-a\sqrt{b^2+x^2}} \cos(cx) dx = \frac{ab}{\sqrt{a^2+c^2}} K_1(b\sqrt{a^2+c^2}), \quad (\text{F.34})$$

where a, b, c are real constants here, and putting $c = 0$, Π_{30}^5 becomes,

$$\Pi_{30}^5 = \frac{e^3 \mathcal{B}}{\pi^2} m \beta \cosh(\beta\mu) K_1(m\beta). \quad (\text{F.35})$$

$K_1(m\beta)$ is the modified Bessel function.

APPENDIX G

Gauge invariance

The axialvector-vector amplitude has an electromagnetic vertex and as a result the electromagnetic current must be conserved in that vertex. In our case the ν -vertex is the electromagnetic vertex and the current conservation condition is given in Eq. (4.36). The following sections explicitly verifies the current conservation condition, which is also called the gauge invariance condition here.

G.1 Gauge invariance for $\Pi_{\mu\nu}^5$ to even orders in the external field

The axialvector-vector amplitude even in the external field is given by

$$i\Pi_{\mu\nu}^{5(e)} = -e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(e)}(p, p', s, s'). \quad (\text{G.1})$$

Noting that,

$$q^\alpha p_\alpha = q^{\alpha\parallel} p_{\alpha\parallel} + q^{\alpha\perp} p_{\alpha\perp},$$

we can write Eq.(4.65) as,

$$\begin{aligned} R_{\mu\nu}^{(e)} &\stackrel{\circ}{=} 4i\eta_-(p_0) \left[(\varepsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta - \varepsilon_{\mu\nu\alpha\beta\perp} p^\alpha p'^{\beta\perp} - \varepsilon_{\mu\nu\alpha\perp\beta} p^{\alpha\perp} p'^\beta) (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right. \\ &\quad \left. + \varepsilon_{\mu\nu\alpha\beta\perp} p^\alpha p'^{\beta\perp} \sec^2(e\mathcal{B}s') + \varepsilon_{\mu\nu\alpha\perp\beta} p^{\alpha\perp} p'^\beta \sec^2(e\mathcal{B}s) \right]. \end{aligned} \quad (\text{G.2})$$

Here throughout we have omitted terms such as $\varepsilon_{\mu\nu\alpha\perp\beta\perp} p^{\alpha\perp} p'^{\beta\perp}$, since by the application of Eq. (D.5) we have

$$\begin{aligned} \varepsilon_{\mu\nu\alpha\perp\beta\perp} p^{\alpha\perp} p'^{\beta\perp} &= \varepsilon_{\mu\nu\alpha\perp\beta\perp} p^{\alpha\perp} p'^{\beta\perp} + \varepsilon_{\mu\nu\alpha\perp\beta\perp} p^{\alpha\perp} k^{\beta\perp} \\ &\stackrel{\circ}{=} -\frac{\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s') + \tan(e\mathcal{B}s')} \varepsilon_{\mu\nu\alpha\perp\beta\perp} k^{\alpha\perp} k^{\beta\perp}, \end{aligned}$$

which is zero.

After rearranging the terms appearing in Eq. (G.2), and by the application of Eq. (D.5) and Eq. (D.6) we arrive at the expression

$$\begin{aligned} R_{\mu\nu}^{(e)} \stackrel{\circ}{=} & 4i\eta_-(p_0) \left[\varepsilon_{\mu\nu\alpha\beta} p^\alpha k^\beta (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right. \\ & \left. + \varepsilon_{\mu\nu\alpha\beta_\perp} k^\alpha k^{\beta_\perp} \tan(e\mathcal{B}s) \tan(e\mathcal{B}s') \frac{\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right]. \end{aligned} \quad (\text{G.3})$$

Because of the presence of terms like $\varepsilon_{\mu\nu\alpha\beta} k^\beta$ and $\varepsilon_{\mu\nu\alpha\beta_\perp} k^\alpha$ if we contract $R_{\mu\nu}^{(e)}$ by k^ν , it vanishes.

G.2 Gauge invariance for $\Pi_{\mu\nu}^5$ to odd orders in the external field

The axialvector-vector amplitude odd in the external field is given by

$$i\Pi_{\mu\nu}^{5(o)} = -e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(o)}(p, p', s, s'), \quad (\text{G.4})$$

where $R_{\mu\nu}^{(o)}(p, p', s, s')$ is given by Eq. (4.66). The general gauge invariance condition in this case

$$k^\nu \Pi_{\mu\nu}^{5(o)} = 0, \quad (\text{G.5})$$

can always be written down in terms of the following two equations,

$$k^\nu \Pi_{\mu\parallel\nu}^{5(o)} = 0, \quad (\text{G.6})$$

$$k^\nu \Pi_{\mu\perp\nu}^{5(o)} = 0, \quad (\text{G.7})$$

where $\Pi_{\mu\parallel\nu}^{5(o)}$ is that part of $\Pi_{\mu\nu}^{5(o)}$ where the index μ can take the values 0 and 3 only. Similarly $\Pi_{\mu\perp\nu}^{5(o)}$ stands for the part of $\Pi_{\mu\nu}^{5(o)}$ where μ can take the values 1 and 2 only. $\Pi_{\mu\parallel\nu}^{5(o)}$ contains $R_{\mu\parallel\nu}^{(o)}(p, p', s, s')$ which from Eq. (4.66) is as follows,

$$\begin{aligned} R_{\mu\parallel\nu}^{(o)} \stackrel{\circ}{=} & 4i\eta_+(p_0) \left[-\varepsilon_{\mu\parallel\nu 12} \left\{ \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_\perp^2 + (k \cdot p)_\parallel (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \right\} \right. \\ & + 2\varepsilon_{\mu\parallel 12\alpha_\parallel} (p'_{\nu\parallel} p^{\alpha\parallel} \tan(e\mathcal{B}s) + p_{\nu\parallel} p'^{\alpha\parallel} \tan(e\mathcal{B}s')) \\ & + g_{\mu\parallel\alpha_\parallel} k_{\nu\perp} \left\{ p^{\tilde{\alpha}\parallel} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) - k^{\tilde{\alpha}\parallel} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right\} \\ & \left. + g_{\mu\parallel\nu} (p \cdot \tilde{k})_\parallel (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \right], \end{aligned} \quad (\text{G.8})$$

and $\Pi_{\mu\perp\nu}^{5(o)}$ contains $R_{\mu\perp\nu}^{(o)}(p, p', s, s')$ which is

$$\begin{aligned} R_{\mu\perp\nu}^{(o)} \stackrel{\circ}{=} & 4i\eta_+(p_0) \left[\{g_{\mu\perp\nu}(p \cdot \tilde{k})_\parallel + g_{\nu\alpha_\parallel} p^{\tilde{\alpha}\parallel} k_{\mu\perp}\} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \right. \\ & \left. + g_{\nu\alpha_\parallel} k^{\tilde{\alpha}\parallel} p_{\mu\perp} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right]. \end{aligned} \quad (\text{G.9})$$

Eq. (G.6) and Eq. (G.7) implies one should have the following relations satisfied,

$$k^\nu \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\perp\nu}^{(o)} = 0, \quad (\text{G.10})$$

and

$$k^\nu \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\parallel\nu}^{(o)} = 0. \quad (\text{G.11})$$

Out of the two above equations, Eq. (G.10) can be verified easily since

$$k^\nu R_{\mu\perp\nu} = 0. \quad (\text{G.12})$$

Now we look at Eq. (G.11). We explicitly consider the case $\mu_\parallel = 3$ (the $\mu_\parallel = 0$ case lead to similar result). For $\mu_\parallel = 3$

$$k^\nu R_{3\nu}^{(o)} \stackrel{\circ}{=} -p_0 \left[(p'^2 - p_\parallel^2)(\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) - k_\perp^2(\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \right] (4i\eta_+(p_0)). \quad (\text{G.13})$$

Apart from the small convergence factors,

$$\begin{aligned} \frac{i}{e\mathcal{B}} (\Phi(p,s) + \Phi(p',s')) &= (p'^2 + p_\parallel^2 - 2m^2) \xi - (p'^2 - p_\parallel^2) \zeta - p_\perp'^2 \tan(\xi - \zeta) \\ &\quad - p_\perp^2 \tan(\xi + \zeta), \end{aligned} \quad (\text{G.14})$$

where we have defined the parameters

$$\begin{aligned} \xi &= \frac{1}{2}e\mathcal{B}(s + s'), \\ \zeta &= \frac{1}{2}e\mathcal{B}(s - s'). \end{aligned} \quad (\text{G.15})$$

From the last two equations we can write

$$ie\mathcal{B} \frac{d}{d\zeta} e^{\Phi(p,s)+\Phi(p',s')} = e^{\Phi(p,s)+\Phi(p',s')} \left(p'^2 - p_\parallel^2 - p_\perp'^2 \sec^2(\xi - \zeta) + p_\perp^2 \sec^2(\xi + \zeta) \right), \quad (\text{G.16})$$

which implies

$$p'^2 - p_\parallel^2 \stackrel{\circ}{=} ie\mathcal{B} \frac{d}{d\zeta} + \left[p_\perp'^2 \sec^2(e\mathcal{B}s') - p_\perp^2 \sec^2(e\mathcal{B}s) \right]. \quad (\text{G.17})$$

The equation above is valid in the sense that both sides of it actually acts upon $e^{\tilde{\Phi}(p,s,p',s')}$ inside the momentum integrals, where

$$\tilde{\Phi}(p, p', s, s') = \Phi(p, s) + \Phi(p', s'). \quad (\text{G.18})$$

From Eq. (G.13) and Eq. (G.17) we have

$$\begin{aligned} k^\nu R_{3\nu} &\stackrel{\circ}{=} -4i\eta_+(p_0)p_0 \left[(p'^2_\perp \sec^2(e\mathcal{B}s') - p^2_\perp \sec^2(e\mathcal{B}s))(\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \right. \\ &\quad \left. - k_\perp^2(\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) + ie\mathcal{B}(\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \frac{d}{d\zeta} \right]. \end{aligned} \quad (\text{G.19})$$

Now using the the expressions for p_\perp^2 and p'^2_\perp from Eq. (D.8) and Eq. (D.9) we can write

$$\begin{aligned} k^\nu R_{3\nu} &\stackrel{\circ}{=} 4e\mathcal{B}\eta_+(p_0)p_0 \left[(\sec^2(e\mathcal{B}s) - \sec^2(e\mathcal{B}s')) \right. \\ &\quad \left. + (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \frac{d}{d\zeta} \right], \end{aligned} \quad (\text{G.20})$$

Explicitly writing $e^{\tilde{\Phi}}$ in its proper place, the above equation can also be written as

$$k^\nu R_{3\nu} e^{\tilde{\Phi}} = 4e\mathcal{B}\eta_+(p_0)p_0 \frac{d}{d\zeta} \left[e^{\tilde{\Phi}}(\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \right]. \quad (\text{G.21})$$

Transforming to ξ, ζ variables and using the above equation we can write the parametric integrations (integrations over s and s') on the left hand side of Eq. (G.11) as

$$\int_{-\infty}^{\infty} ds \int_0^{\infty} ds' k^\nu R_{3\nu} e^{\tilde{\Phi}} = \frac{8\eta_+(p_0)p_0}{e\mathcal{B}} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi - \zeta) \frac{d}{d\zeta} \mathcal{F}(\xi, \zeta) \quad (\text{G.22})$$

where

$$\mathcal{F}(\xi, \zeta) = e^{\tilde{\Phi}}(\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')). \quad (\text{G.23})$$

The integration over the ξ and ζ variables in Eq. (G.22) can be represented as,

$$\begin{aligned} &\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi - \zeta) \frac{d}{d\zeta} \mathcal{F}(\xi, \zeta) \\ &= \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\zeta \left[\frac{d}{d\zeta} \{ \Theta(\xi - \zeta) \mathcal{F}(\xi, \zeta) \} - \delta(\xi - \zeta) \mathcal{F}(\xi, \zeta) \right] \\ &= - \int_{-\infty}^{\infty} d\xi \mathcal{F}(\xi, \xi) \end{aligned} \quad (\text{G.24})$$

here the second step follows from the first one as the first integrand containing the Θ function vanishes at both limits of the integration. The remaining integral is now only a function of ξ and is even in p_0 . But in Eq. (G.22) we have $\eta_+(p_0)p_0$ sitting, which makes the the integrand odd under p_0 integration in the left hand side of Eq. (G.11), as $\eta_+(p_0)$ is an even function in p_0 . So the p_0 integral as it occurs in the left hand side of Eq. (G.11) vanishes as expected, yielding the required result shown in Eq. (G.6).

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